Numerical Analysis and Adaptive Computation for Solutions of Elliptic Problems with Randomly Perturbed Coefficients

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We develop a reliable efficient method for computing solutions to the Poisson equation a with randomly perturbed coefficient. We assume the perturbation to be piecewise constant and use a non-overlapping domain decomposition algorithm, where the domains coincides with regions where the perturbation is constant, to solve the equations. On each sub-domain we use an truncated Neumann series to approximate the inverse of the local stiffness matrix. By doing so we can solve for all samples simultaneously in a very efficient way. We derive a posteriori error estimates and construct an adaptive algorithm to tune the method parameters automatically.

1 Introduction

For practical application of simulation of physical processes using computational methods, experimental data that is very likely to include measurement errors, must be used in the simulation. It is important to understand the sensitivity of the computed solution to these errors in order to attain reliable results. In this paper, we model these errors as random perturbations in the coefficients. We aim to develop a method for cheaply computing a large number of realizations of the solution in order to compute of a distribution in a quantity of interest computed from the solution including the effect of the associated errors. The Poisson equation is considered with randomly perturbed diffusion coefficient.

There have been a lot of work on solving partial differential equations with random coefficients recently, see [1] and references therein. The standard Monte Carlo Finite Element Method e.g. approximates the expected value of the solution pointwise in space by taking sample averages of independent identically distributed data. In [2] we present an efficient method for computing solutions to the Poisson equation with a randomly perturbed coefficient in the case when the random perturbation can be described as a piecewise constant function. The method uses a non-overlapping domain decomposition method developed by Lions' together with the ideas of the standard Monte Carlo Finite Element Method as a starting point.

The key idea of our method is that if the random perturbation is a constant random number on each of the domains in the domain decomposition algorithm we can compute solutions corresponding to a vast number of realizations of the diffusion coefficient simultaneously by using the summation formula for the inverse of a perturbed identity matrix. We present error estimates taking into account not only the discretization error and the stochastic error due to finite sample size, but also the number of iterations in the domain decomposition algorithm, and the number of terms used in the truncated summation representation of the inverse of the perturbed identity matrix. The error estimate is then used to derive an adaptive algorithm that automatically tunes the method parameters.

2 The method

We study the Poisson equation with randomly perturbed coefficient on weak form, find $\{u_n\}_{n=1}^N \in H_0^1(\Omega)$ such that,

$$((a+\delta_n)\nabla u^n, \nabla v) = (f, v) \quad \text{for all } v \in H_0^1(\Omega), \tag{1}$$

given perturbations $\{\delta_n\}_{n=1}^N$ and deterministic functions a and f. We discretize the problem by introducing a finite element space $V_h \subset H_0^1(\Omega)$ and introduce corresponding discrete solutions $\{u_h^n\}_{n=1}^N$. We solve the problem using Lions' non-overlapping domain decomposition algorithm, see [3]. On each domain Ω_d, $\cup\Omega_d = \Omega$, we need to solve local systems of equations of the following form,

$$(A + \delta^n K)u_h^{n,d} = b^n, \tag{2}$$

where A corresponds to the bilinear form $(a\nabla v, \nabla w)_{\Omega_d} + \lambda(v,w)_{\partial\Omega_d}$, where $\lambda(v,w)_d$ represents boundary terms, K corresponds to the bilinear form $(\nabla v, \nabla w)_{\Omega_d}$, b_n corresponds to right hand side load f and boundary contribution from neighboring

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domains computed in previous iterations, and $u_h^{n,d}$ is u_h^n restricted to domain Ω_d. We manipulate equation (2) in the following way,

$$u_h^{n,d} = (I + \delta^n A^{-1} K)^{-1} A^{-1} b_n, \tag{3}$$

where I is the identity matrix and A is assumed to be invertible. We now solve the local problems approximately by replacing the inverse with a truncated Neumann series of P terms,

$$u_{h,P}^{n,d} = \sum_{p=0}^{P-1} (-\delta^n A^{-1} K)^p A^{-1} b_n, \tag{4}$$

and we show that the sum converges as $P \to \infty$, see [2]. The gain in doing this is that the number of local systems that have to be solved becomes independent of N. The total approximate solution $u_{h,P,J}^n$ depends on mesh size h, number of iterations J in the domain decomposition algorithm, and number of terms in the approximation of the perturbed identity matrix P.

3 Error estimation

Given a particular realization δ^n, we approximate the error in a linear functional $(e^n, \psi) = (u^n - u_{h,J,P}^n, \psi)$ by solving the corresponding randomly perturbed adjoint problem, find $\{\phi_n\}_{n=1}^N \in H_0^1(\Omega)$ such that,

$$((a + \delta^n)\nabla\phi^n, \nabla v) = (\psi, v), \quad \text{for all } v \in H_0^1(\Omega), \tag{5}$$

approximately using the same method and constructing the error representation formula,

$$(e^n, \psi) = (f, \phi^n) - ((a + \delta^n)\nabla u_{h,J,P}^n, \nabla\phi^n). \tag{6}$$

By equation (6) we can approximate the error of a linear functional of each single realization δ^n. Given this error representation formula we can also easily derive indicators that bounds components of the error associated with h, J, and P, i.e.,

$$|(e^n, \psi)| \le |(e_h^n, \psi)| + |(e_J^n, \psi)| + |(e_P^n, \psi)| = e_\psi^n \tag{7}$$

see [2] for complete description of these terms. This completes the analysis for a single realization but the goal is of course to get an estimate of the error for a stochastic quantity such as the distribution function of the error, i.e. $F(x) - \tilde{F}_{f,J,P}^N$ where $F(x) = P(\{(u^n, \psi)\} < x)$ and $\tilde{F}_{h,J,P}^N = P(\{(U_{h,J,P}^n, \psi)\}_{n=1}^N < x)$. Here $\tilde{F}$ is computed using N samples of the coefficient δ^n while the exact distribution is computed using the entire distribution. We present the following theorem where the error in the distribution function is bounded in terms of the individual error indicators for each sample and the number of samples N.

Theorem 3.1 *We let $e_M = \sup_n e_\psi^n$ and assume that N is large enough for the Central Limit Theorem to be valid. Then,*

$$|F(x) - \tilde{F}_{h,J,P}^N(x)| \le \tau \sqrt{\frac{\tilde{F}_{h,J,P}^N(x)(1 - \tilde{F}_{h,J,P}^N(x))}{N}} + e_M \frac{\partial}{\partial x} \tilde{F}_{h,J,P}^N(x), \tag{8}$$

holds with approximate probability $\int_{-\infty}^{\tau} e^{-t^2/2}\, dt/\sqrt{2\pi}$.

We get one term that depends on the number of samples N and one terms that depends on the numerical error bound e^M which in turn depends on the parameters used to compute each single realization, h, J, and P. Note that the right hand side in equation (8) is computable.

This estimate can then be used to construct an adaptive algorithm that automatically tunes the parameters N, J, P, and h in order to minimize the error in the distribution function of the linear functional. Note that e_M by equation (7) can be controlled by decreasing h, J, and P.

Remark 3.2 Since non-overlapping domain decomposition algorithms converges slowly when the number of sub domains increases it is very natural to add coarse grid correction to the algorithm. This is possible and on the coarse scale on truncation is used. If the coefficient a is continuous, it may be unwise to add a piecewise constant perturbation since that will introduce artificial loss of regularity. This can be avoided by using continuous piecewise linear random perturbations. The inverses of the local stiffness matrices do not have to be formed explicitly.

References

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