The Shapley-Shubik Power Index

Differs from Banzhaf Power Index:

✦✦ order of the players is important ✦✦

Who joined the coalition first?

Example: Under the Banzhaf method, $\{P_1, P_2, P_3\}$ is the same as $\{P_3, P_1, P_2\}$. Under Shapley-Shubik, these are different coalitions.

Change in notation:

Use $\langle P_1, P_2, P_3 \rangle$ for sequential coalition

Under Banzhaf, we count all sizes of coalitions.

Under Shapley-Shubik, we count only coalitions of size N.

One ordinary coalition of 3 players, $\{P_1, P_2, P_3\}$, has 6 sequential coalitions:

$\langle P_1, P_2, P_3 \rangle$, $\langle P_1, P_3, P_2 \rangle$, $\langle P_2, P_1, P_3 \rangle$, $\langle P_3, P_2, P_1 \rangle$, $\langle P_2, P_3, P_1 \rangle$, $\langle P_3, P_1, P_2 \rangle$.

How many sequential coalitions are there for N players?

If $N = 3$ there are 6 coalitions. What if $N = 4$?

$$\Big\langle \underset{4}{\underline{\quad}}, \underset{3}{\underline{\quad}}, \underset{2}{\underline{\quad}}, \underset{1}{\underline{\quad}} \Big\rangle$$

Total number of sequential coalitions:

$4 \times 3 \times 2 \times 1 = 24$.

Reason this way: We have 4 choices to fill the first slot, then only 3 for the second slot. The third slots has 2 choices and there is only one choice left for the last slot.

Multiplication Rule: ***If there are m different ways to do X and n different ways to do Y, then X and Y together can be done in*** $m \times n$ ***different ways.***

Ice Cream Example: 2 cone types, 3 flavors means 6 possible cone/flavor combinations.

Back to the number of coalitions: if N=5,
$1 \times 2 \times 3 \times 4 \times 5 = 120$ **coalitions.**

Notation: **factorial (!)**
$1 \times 2 \times 3 \times 4 \times \cdots \times N = N!$
If there are N players, there are N! sequential coalitions.

Shapley-Shubik Power Index:
Look for the pivotal player:
the player that tips the scales and turns a losing coalition into a winning coalition.

Note- there is one and only one pivotal player in any coalition.

Start at first player and ask: Are there enough votes?

Add the second player, ask: Are there enough votes?

Continue until the answer is YES! -this is the pivotal player

Consider: The players that were part of the coalition before the pivotal player did not have enough votes to carry a motion and the players that come after the pivotal player don't really matter.

Shapley-Shubik depends on the total number of times that a player is pivotal in relation to all other players.

Finding the Shapley-Shubik Power Index

- ✧ **Make a list of all sequential coalitions containing all N players. (There are $N!$ of them.)**
- ✧ **In each sequential coalition determine *the* pivotal player. (There is one in each sequential coalition.)**

The Shapley-Shubik power index of a player is given by:

$$\frac{\text{The number of times the player is pivotal}}{N!}$$

Example:

$[4 : 3, 2, 1]$ **There are** $3! = 6$ **sequential coalitions:**

$\langle P_1, P_2, P_3\rangle \quad \langle P_1, P_3, P_2\rangle \quad \langle P_2, P_1, P_3\rangle$

$\langle P_3, P_2, P_1\rangle \quad \langle P_2, P_3, P_1\rangle \quad \langle P_3, P_1, P_2\rangle$

Find the pivotal players.

How many times is each player pivotal?

P_1 **is pivotal 4 times.**

P_2 **is pivotal 1 times.**

P_3 **is pivotal 1 times.**

The Shapley-Shubik power distribution is:

$P_1 : \frac{4}{6} = \frac{2}{3} = 66\frac{2}{3}\%$, $P_2 : \frac{1}{6} = 16\frac{2}{3}\%$, $P_3 : \frac{1}{6} = 16\frac{2}{3}\%$

Note results from Banzhaf distribution:

$P_1 : 60\%$, $P_2 : 20\%$, $P_3 : 20\%$

Example: Akron Flyers Basketball Draft

Note: use weights instead of player names.

$\langle 4, \underline{3}, 2, 1\rangle$	$\langle 3, \underline{4}, 2, 1\rangle$	$\langle 2, \underline{4}, 3, 1\rangle$
$\langle 1, 4, \underline{3}, 2\rangle$	$\langle 4, \underline{3}, 1, 2\rangle$	$\langle 3, \underline{4}, 1, 2\rangle$
$\langle 2, \underline{4}, 1, 3\rangle$	$\langle 1, 4, \underline{2}, 3\rangle$	$\langle 4, \underline{2}, 3, 1\rangle$
$\langle 3, 2, \underline{4}, 1\rangle$	$\langle 2, 3, \underline{4}, 1\rangle$	$\langle 1, 3, \underline{4}, 2\rangle$
$\langle 4, \underline{2}, 1, 3\rangle$	$\langle 3, 2, \underline{1}, 4\rangle$	$\langle 2, 3, \underline{1}, 4\rangle$
$\langle 1, 3, \underline{2}, 4\rangle$	$\langle 4, 1, \underline{3}, 2\rangle$	$\langle 3, 1, \underline{4}, 2\rangle$
$\langle 2, 1, \underline{4}, 3\rangle$	$\langle 1, 2, \underline{4}, 3\rangle$	$\langle 4, 1, \underline{2}, 3\rangle$
$\langle 3, 1, \underline{2}, 4\rangle$	$\langle 2, 1, \underline{3}, 4\rangle$	$\langle 1, 2, \underline{3}, 4\rangle$

The Shapley-Shubik power distribution is:

Coach: $\frac{10}{24} = \frac{5}{12} = 41\frac{2}{3}\%$

Manager: $\frac{6}{24} = \frac{1}{4} = 25\%$

Scout: $\frac{6}{24} = \frac{1}{4} = 25\%$

Trainer: $\frac{2}{24} = \frac{1}{12} = 8\frac{1}{3}\%$

Note: same as Banzhaf power distribution.

Example: City of Cleansburg

Council with 1 mayor and 4 "ordinary" members.

A motion passes if the mayor and 2 others vote yes, or if all 4 ordinary members vote yes. (The mayor has veto power but a unanimous vote can override the mayor's veto.)

There are $5! = 120$ sequential coalitions to consider!

Try a short cut.

In what position(s) is the mayor pivotal?

$$\langle \_\_\_, \_\_\_, \_\_\_, \_\_\_, \_\_\_ \rangle$$

The mayor is pivotal in 3rd and 4th positions.

How many sequential coalitions are there with the mayor in 3rd and 4th place?

Is the number of coalitions with the mayor in 3rd place the same as the number of coalitions with the mayor in 4th place? How about the number of coalitions with the mayor in 1st place? 2nd place? etc.

Note: there are 5 positions that the mayor can be in and for each position, there are the same number of coalitions. This makes 5 groups of equal size. Thus we can divide the total number of coalitions by 5 to get the number of coalitions per position.

120 total coalitions/ 5 groups = 24 times mayor in each position
Mayor pivotal in 3rd and 4th, so he is pivotal 48 times.

Mayor's power: $48/120 = 2/5 = 40\%$.

This leaves 60% **of the power to split 4 ways.**
Power of other members: $60\%/4 = 15\%$

Good Exercise for Homework: do the Banzhaf power distribution for this example.

Applications of the Shapley-Shubik Power Index
<u>The Electoral College</u>

51 States (including District of Columbia) means 51! sequential coalitions. this number has 67 digits! In other words, this is all too large to work with, so we'd need to look for mathematical shortcuts and use computers to help.

<u>The United Nations Security Council</u>

5 permanent members, 10 nonpermanent members, total 15 members

15! means about 1.3 trillion sequential coalitions

A nonpermanent member is pivotal only if it is the 9^{th} **player in the coalition, preceded by all five of the permanent members and three nonpermanent members. This happens in about 2.44 billion sequential coalitions.**

Nonpermanent member has a Shapley-Shubik index of 2.44 billion/1.3 trillion or 0.19%

Divide the rest of the 98% **of power among 5 permanent members to get a Shapley-Shubik power index of** 19.6% **for a permanent member.**

Note that with large N**'s we need to use reasoning, approximation and computers rather than finding the power distribution by hand.**

Power Distribution Summary

- **Banzhaf Power Distribution**

  - ***critical* players (can have more than 1 per coalition)**
  - **uses** $\{P_1, P_2\}$ **notation**
  - 2^N **coalitions**
  - **use coalitions from empty set to grand coalition**

- **Shapley-Shubik Power Distribution**

  - ***pivotal* players (only one per sequential coalition)**
  - **uses** $\langle P_1, P_2 \rangle$ **notation**
  - $N!$ **sequential coalitions**
  - **all sequential coalitions the same size**