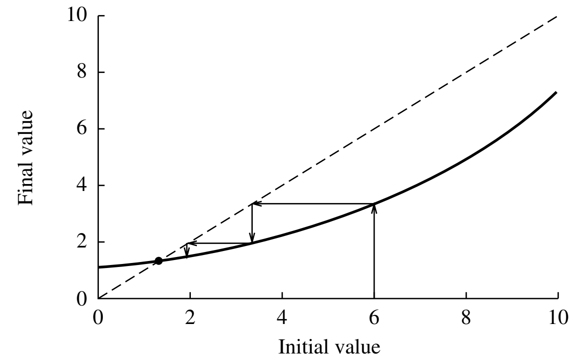
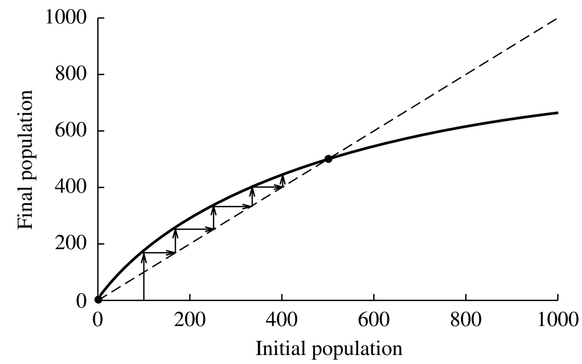
Chapter 3

Section 3.1, page 248

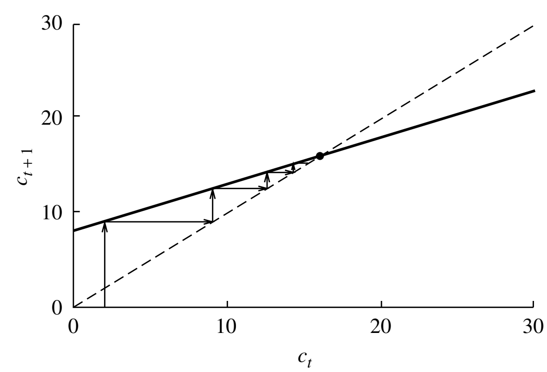
1. The updating function crosses from above to below and is stable.



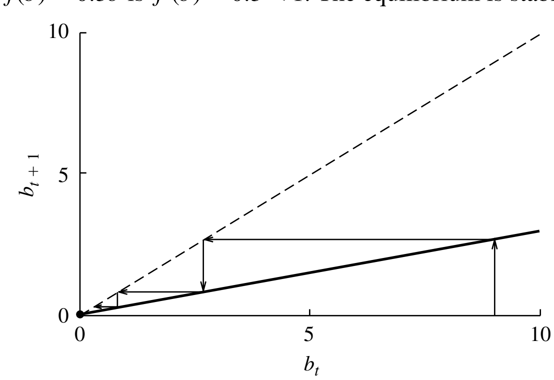
3. At the lower equilibrium, the updating function crosses the diagonal from below to above and is therefore unstable. At the upper equilibrium, the updating function crosses from above to below and is stable.



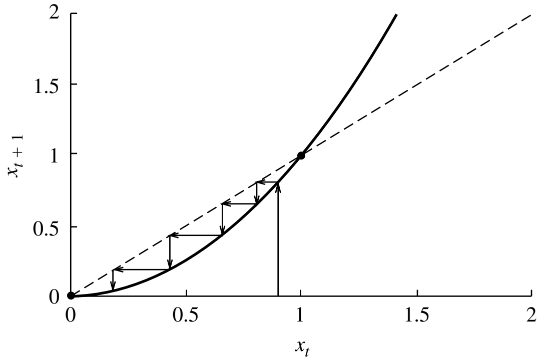
5. The equilibrium satisfies $c^* = 0.5c^* + 8.0$ or $c^* = 16.0$. The slope of the updating function $f(c) = 0.5c + 8.0$ is $f'(c) = 0.5 < 1$. The equilibrium is stable.



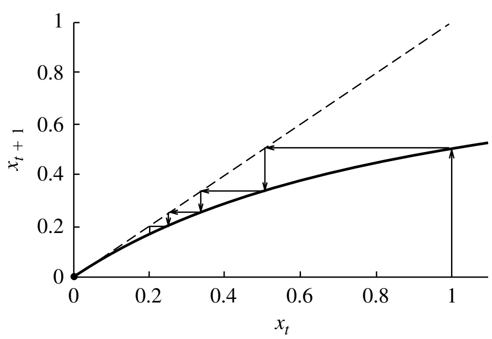
7. The equilibrium is $b^* = 0$. The slope of the updating function $f(b) = 0.3b$ is $f'(b) = 0.3 < 1$. The equilibrium is stable.



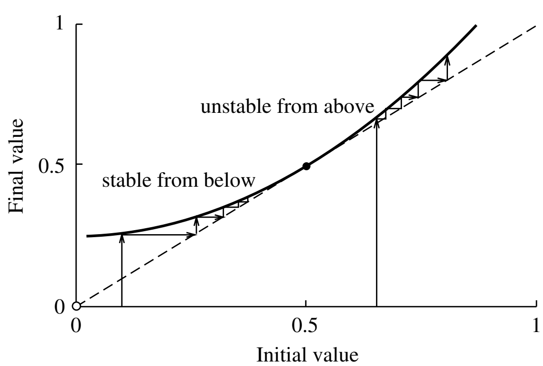
9. $f(x) = x$ when $x^2 = x$, or $x^2 - x = 0$, or $x(x-1) = 0$, which has solutions at $x = 0$ and $x = 1$. The derivative is $f'(x) = 2x$, so $f'(0) = 0 < 1$ (stable) and $f'(1) = 2 > 1$ (unstable).



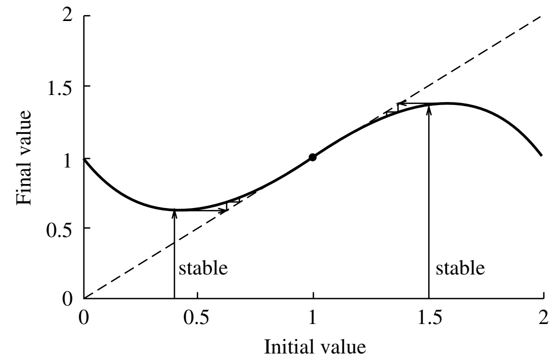
11. $x^* = \dfrac{x^*}{1+x^*}$ has solution $x^* = 0$. Also, if $f(x) = \dfrac{x}{1+x}$, then $f'(x) = \dfrac{1}{(1+x)^2}$, so $f'(0) = 1$. We can't tell whether this is stable or not using the Slope Criterion for stability. However, as in Example 3.1.9, the graph of the updating function lies below the diagonal for all $x > 0$, meaning that the equilibrium is stable.



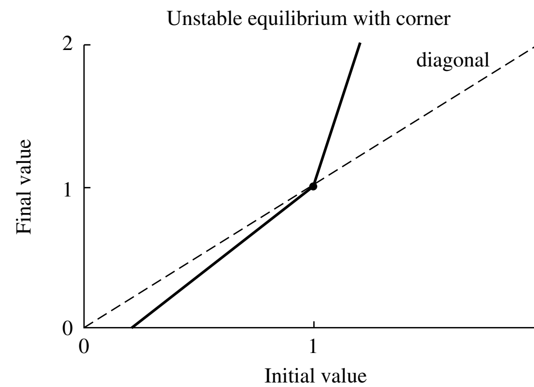
13.



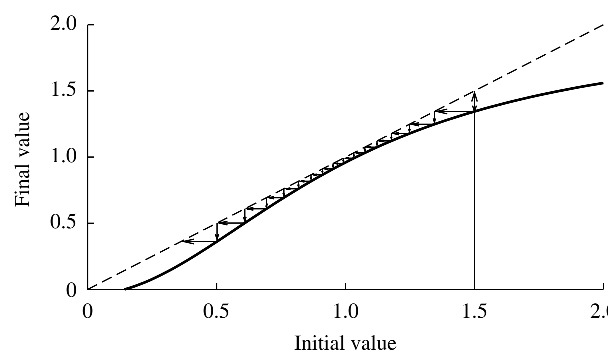
15. The second derivative is 0 at the equilibrium.



17.



19. There is no equilibrium, and the cobwebbing creeps slowly past the point where the equilibrium used to be.

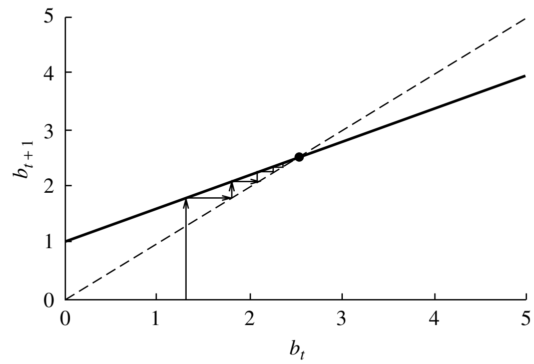


21. The inverse is $f^{-1}(x) = \dfrac{x}{1-x}$. The only equilibrium is at $x = 0$. The slope of both the original updating function and the inverse is 1 at this point.

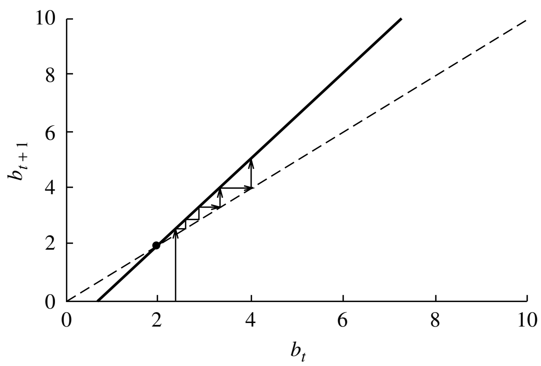
23. $f'(p) = \dfrac{2.4}{[1.2p + 2.0(1-p)]^2}$. $f'(0) = 0.6$, $f'(1) \approx 1.667$. The equilibrium at $p = 0$ is stable; the equilibrium at $p = 1$ is unstable.

25. $f'(p) = \dfrac{2.25}{[1.5p + 1.5(1-p)]^2} = 1$. $f'(0) = f'(1) = 1$. Both derivatives are exactly 1, so we cannot tell. This updating function exactly matches to diagonal, meaning that solutions move neither toward nor away from equilibria.

27. The discrete-time dynamical system is $b_{t+1} = 0.6b_t + 1.0 \times 10^6$. The equilibrium satisfies $b^* = 0.6b^* + 1.0 \times 10^6$, or $0.4b^* = 1.0 \times 10^6$, or $b^* = 2.5 \times 10^6$. The derivative of the updating function $f(b) = 0.6b + 1.0 \times 10^6$ is $f'(b) = 0.6$, which is always less than 1. The equilibrium is stable.



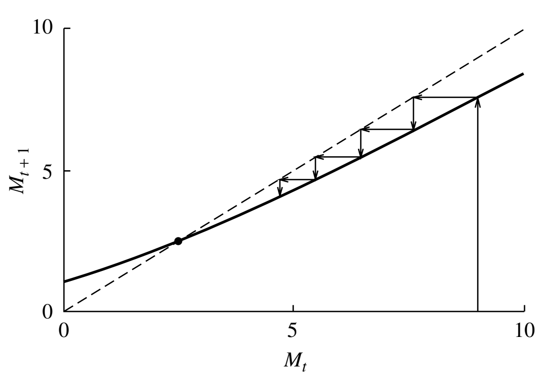
29. The updating function is $b_{t+1} = 1.5b_t - 1.0 \times 10^6$, and the equilibrium is $b^* = 2.0 \times 10^6$. The derivative of the updating function is $f'(b) = 1.5 > 1$, so the equilibrium is unstable.



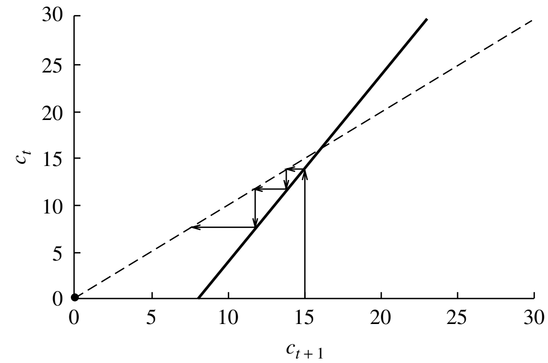
31. The equilibrium concentration of medication is 2.5. The derivative of the updating function is

$$f'(M) = 1 - \frac{0.5}{(1 + 0.1M)^2}$$

so $f'(2.5) \approx 0.68$. The equilibrium is stable.



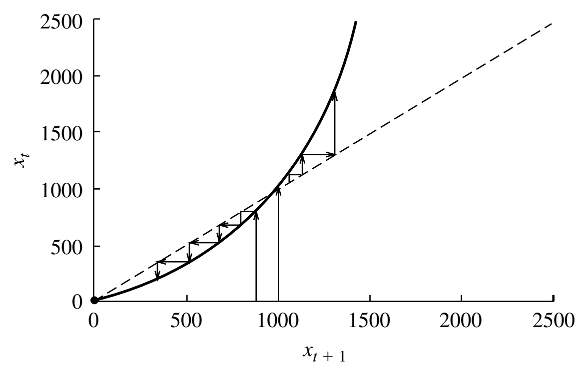
33. The inverse is $c_t = 2.0(c_{t+1} - 8.0) = f^{-1}(c_{t+1})$. The equilibrium is at 16.0. The derivative of the backwards updating function is $(f^{-1})'(c) = 2.0$, so the equilibrium is unstable. The same equilibrium was stable in the forward direction.



35. The inverse is

$$x_t = \frac{x_{t+1}}{2 - 0.001x_{t+1}} = f^{-1}(x_{t+1})$$

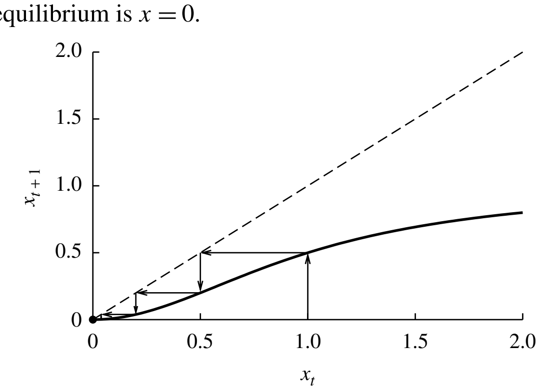
The equilibria are at 0 and 1000. The derivative of the backwards updating function is $(f^{-1})'(x) = \frac{2}{(2 - 0.001x_{t+1})^2}$. Therefore, $(f^{-1})'(0) = 0.5$ and $(f^{-1})'(1000) = 2$. The equilibrium at $x = 0$ is stable and the equilibrium at $x = 1000$ is unstable.



37. $x_{t+1} = \frac{x_t^2}{1.0 + x_t^2}$. The equilibrium satisfies

$$\begin{aligned} x &= \frac{x^2}{1 + x^2} \\ x(1 + x^2) &= x^2 \\ x(1 + x^2 - x) &= 0 \\ x &= 0 \text{ or } (1 + x^2 - x) = 0 \end{aligned}$$

Because $1 + x^2 - x = 0$ has no real solution (the quadratic formula gives the square root of a negative number), the only equilibrium is $x = 0$.

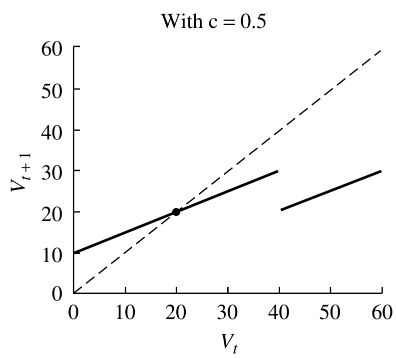


The derivative of the updating function is

$$f'(x) = \frac{2x}{(1+x^2)^2}$$

so $f'(0) = 0$, and the equilibrium is stable. This population is going extinct.

39. The equilibrium is stable because the updating function is a line with slope less than 1.



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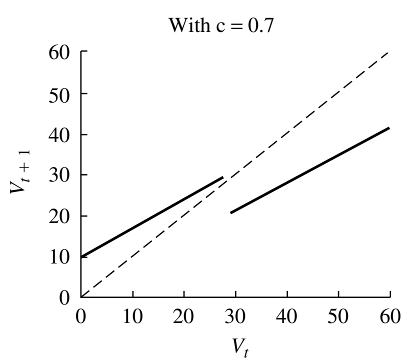
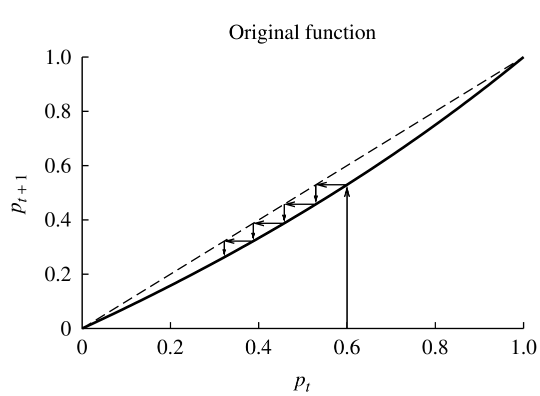
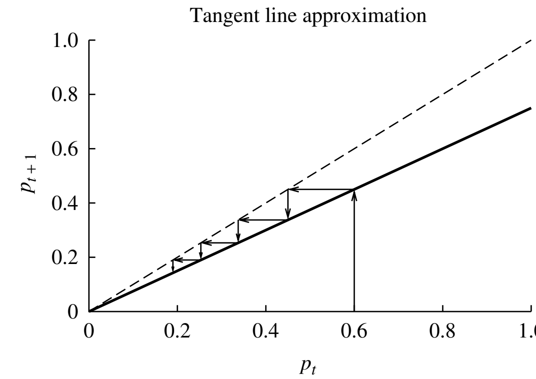


Section 3.2, page 261

1. The derivative of the updating function $f(p) = \dfrac{1.5p}{1.5p + 2.0(1-p)}$ is $f'(p) = \dfrac{3.0}{[1.5p + 2.01 - p)]^2}$ so $f'(0) = 0.75$. The tangent line is $\hat{f}(p) = 0.75p$ and the equilibrium is stable.

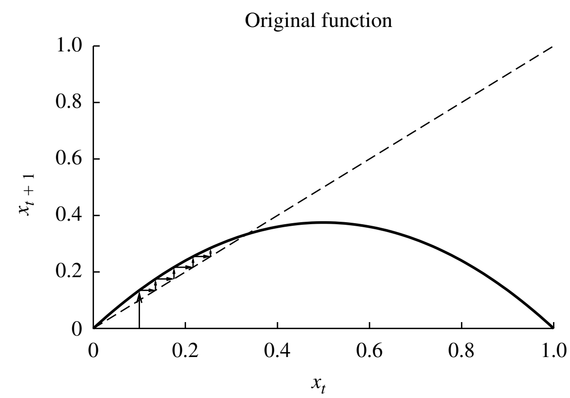


a

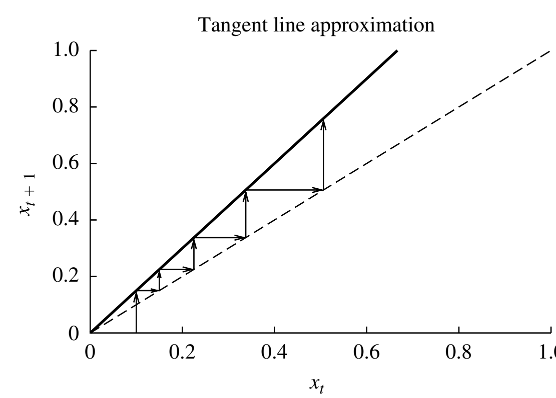


b

3. The derivative of the updating function $f(x) = 1.5x(1-x)$ is $f'(x) = 1.5(1-2x)$, so $f'(0) = 1.5$. The tangent line is $\hat{f}(x) = 1.5x$ and the equilibrium is unstable.



a

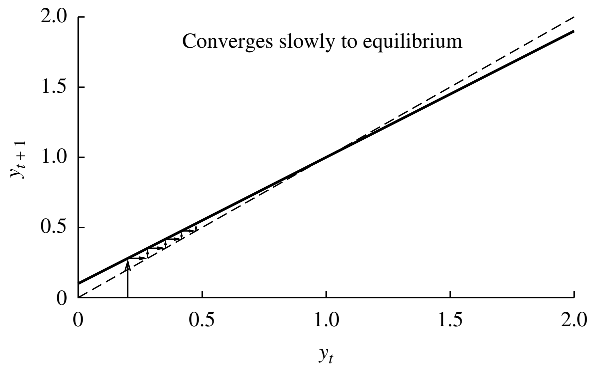


b

5. The solution is $y_t = 2.0 \cdot (1.2)^t$ with values 2.0, 2.4, 2.88, 3.456, 4.147, 4.977. Solving $y_t = 100$ gives $1.2^t = 50$, or $t = \ln(50)/\ln(1.2) = 21.45$. It would cross 100 at time step 22.

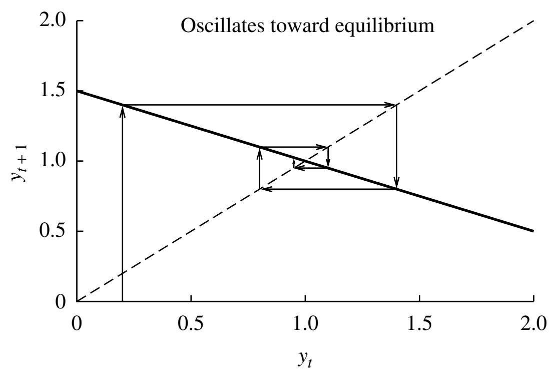
7. The solution is $y_t = 2.0 \cdot (0.8)^t$ with values 2.0, 1.6, 1.28, 1.024, 0.819, 0.655. Solving $y_t = 0.2$ gives $0.8^t = 0.1$, or $t = \ln(0.1)/\ln(0.8) = 10.31$. It would cross 0.2 at time step 11.

9. The equilibrium is $y^* = 1.0$.



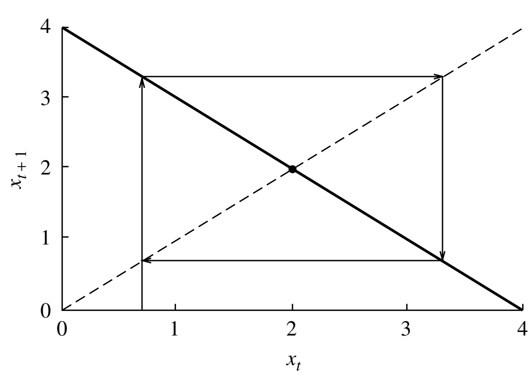
It is stable because the slope m is less than 1.

11. The equilibrium is $y^* = 1.0$.

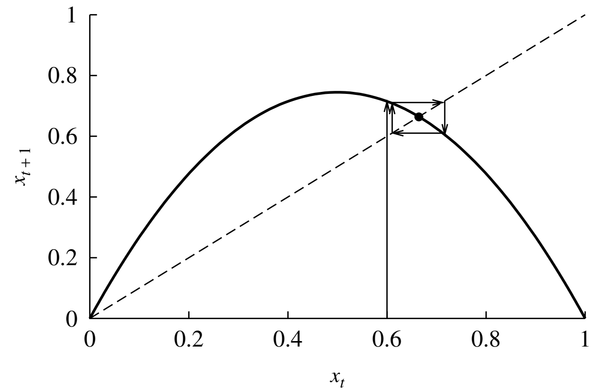


It is stable but oscillatory because $-1 < m < 0$.

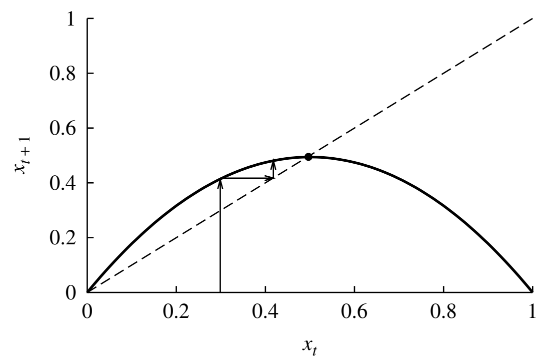
13. The slope of the updating function is exactly -1 everywhere. Solutions jump back and forth and are neither stable nor unstable.



15. $x^* = 3x^*(1 - x^*)$ has solutions $x^* = 0$ and $x^* = 2/3$. The derivative of the updating function $f(x) = 3x(1 - x)$ is $f'(x) = 3(1 - 2x)$, so $f'(0) = 3$ and $f'(2/3) = -1$. The zero equilibrium is unstable, but a solution starting at $x_0 = 0.6$ gets slowly closer to x^*, with $x_2 = 0.6048$ and $x_4 = 0.6087$. The positive equilibrium is stable.



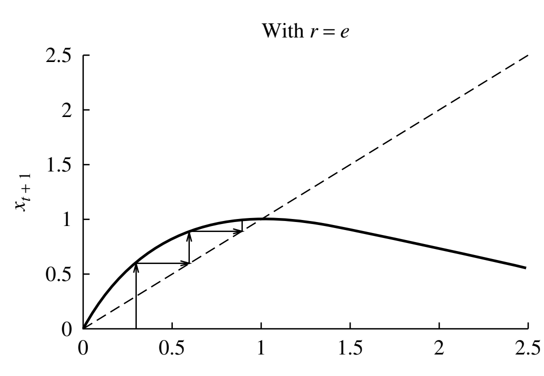
17. $x^* = 0.5$, $f'(x) = 2 - 4x$, and $f'(0.5) = 0$. The equilibrium is highly stable. The solutions of a linear system with slope 0 hit the equilibrium in one step. The solutions in this case move toward the equilibrium very quickly.



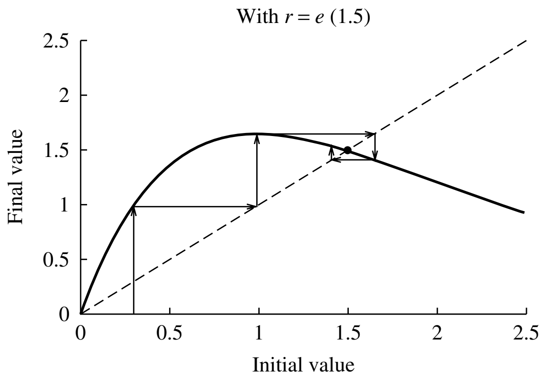
19. The derivative of the updating function $g(M) = M - \dfrac{M^2}{2 + M} + 1$ is $g'(M) = \dfrac{4}{(2 + M)^2}$, so $g'(2) = 1/4$. The equilibrium is stable and does not oscillate.

21. The derivative of the updating function $g(M) = M - \dfrac{M^5}{16 + M^4} + 1$ is $g'(M) = \dfrac{16(3M^4 - 16)}{(16 + M^4)^2}$ and $g'(2) = -1/2$. The equilibrium is stable, but solutions oscillate.

23. The slope at the equilibrium is $1 - \ln(r)$, so it switches sign when $\ln(r) = 1$, or $r = e$.



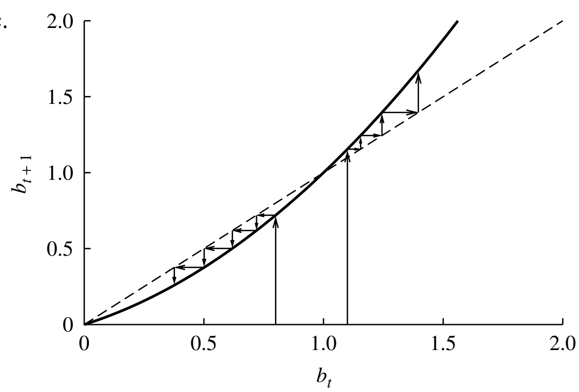
25. I chose the value $r = e^{1.5} \approx 4.482$.



27. a. The discrete-time dynamical system is $b_{t+1} = b_t(0.5 + 0.5b_t)$.

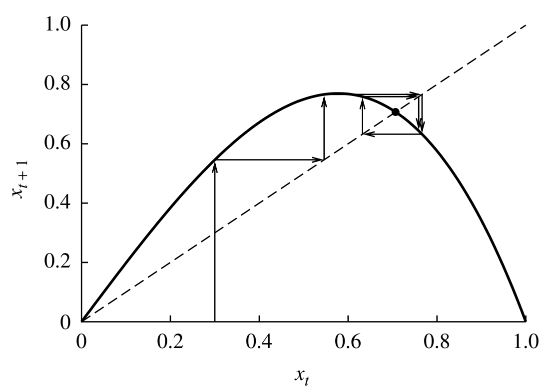
b. The equilibria are at $b^* = 0$ and $b^* = 1$. The derivative of the updating function $f(b) = (0.5 + 0.5b)b$ is $f'(b) = 0.5 + b$, so $f'(0) = 0.5$ and $f'(1) = 1.5$. The equilibrium $b = 0$ is stable and $b = 1$ is unstable.

c.



d. Populations starting below 1 die out, whereas those starting above 1 blast off to infinity. This species does well with a little help from its friends.

29. a.



b. The equilibria are at $x = 0$ and $x^* = \sqrt{1 - \frac{1}{r}}$ (when $r \geq 1$).

c. $f'(x) = r - 3rx^2$. $f'(0) = r$ and $f'(x^*) = 3 - 2r$.

d. The first equilibrium is stable when $r < 1$ as usual, and the positive equilibrium is stable when $r < 2$.

31. a. This line passes through the points (20, 21) and (19, 18). Let z be the setting on the thermostat, and let T be the temperature. The equation is $T = 3(z - 20) + 21$.

b. When it is 18°C, you set the thermometer to 22°C. This results in a temperature of 27°C. Setting the thermometer to 13°C then results in a temperature of 0°C. Things are getting pretty chilly.

c. $z_t = 40 - T_t$.

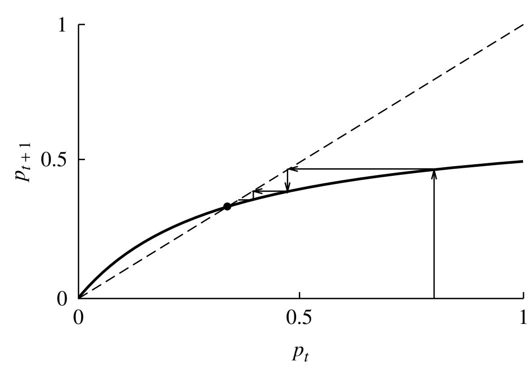
d. $T_{t+1} = 3(z_t - 20) + 21 = 3(40 - T_t - 20) + 21 = 81 - 3T_t$.

e. The slope of −3 means that the temperatures will oscillate more and more widely. The system needs a better correction mechanism. I would have it estimate T as a function of z and correct on that basis.

33. The mutants do worse when mutants are common, and the wild type do better. The discrete-time dynamical system is

$$\begin{aligned} p_{t+1} &= \frac{a_{t+1}}{a_{t+1} + b_{t+1}} \\ &= \frac{2(1 - p_t)a_t}{2(1 - p_t)a_t + (1 + p_t)b_t} \\ &= \frac{2(1 - p_t)p_t}{2(1 - p_t)p_t + (1 + p_t)(1 - p_t)} \\ &= \frac{2p_t}{2p_t + (1 + p_t)} \end{aligned}$$

The equilibria are at $p = 0$ and $p = 1/3$. The derivative of the updating function is $f'(p) = \frac{2}{(1 + 3p)^2}$, so $f'(0) = 2 > 1$ and $f'(1/3) = 1/2 < 1$.

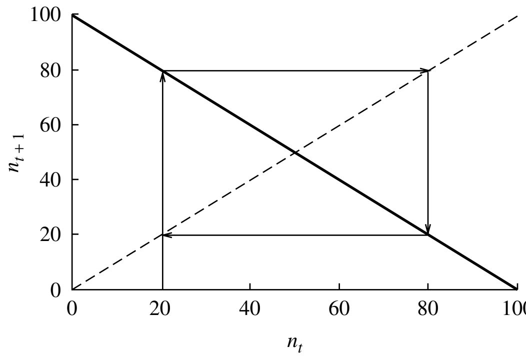


35. a. There are 20 plants, and each grows to size $100/20 = 5$ and makes 4 seeds. This gives making a total of 80. These 80 plants grow to size $\frac{100}{80} = 1.25$, so each makes 0.25 seed (or 1 in 4 plants makes a seed). The total number of seeds is then 20 in the next generation. The values just keep jumping back and forth.

b. There are n_t plants of size $\frac{100}{n_t}$. Each of the n_t plants makes $\frac{100}{n_t} - 1$ seeds, for a total of $n_{t+1} = 100 - n_t$.

c. $n^* = 50$.

d.

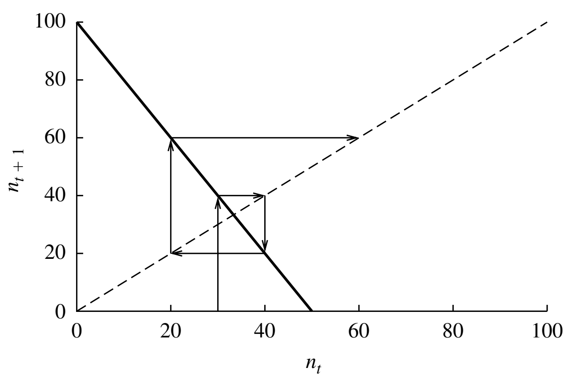


e. The slope is -1, and the equilibrium is neither stable nor unstable.

37. a. There are 20 plants, and each makes 3.0 seeds, for a total of 60. These 60 plants grow to size 1.667, each making a negative number of seeds. This population just went extinct.

b. The discrete-time dynamical system is $n_{t+1} = 100 - 2.0n_t$.

c. There is an equilibrium $n^* \approx 33.3$.



d. The slope is -2.0, and the equilibrium is unstable and oscillatory.

Section 3.3, page 276

1. $a'(x) = \frac{1}{(1+x)^2}$ which is always positive. There are no critical points.

3. $c'(w) = 3w^2 - 3$. Therefore, $c'(w) = 0$ if $3w^2 = 3$ which has solutions at $w = -1$ and $w = 1$.

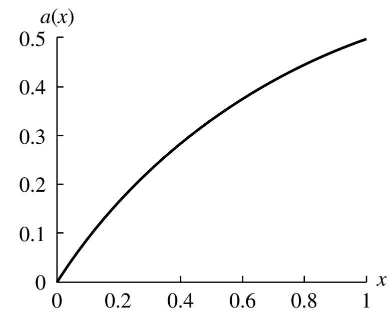
5. $h'(z) = 2ze^{z^2}$, which is 0 only at $z = 0$.

7. There are no critical points. The global maximum is at the endpoint $x = 1$ where $a(1) = 1/2$. The global minimum is at the other endpoint $x = 0$ where $a(0) = 1/2$.

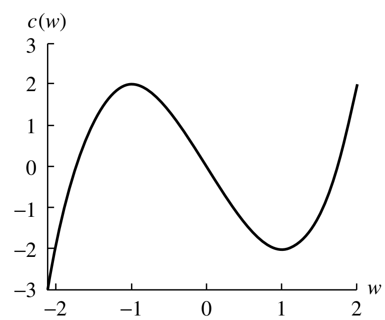
9. The critical points are $x = -1$ and $x = 1$. $c(-1) = 2$ and $c(1) = -2$. At the endpoints, $c(-2) = -2$ and $c(2) = 2$. We have two ties. The global maximum value is 2, taken on at $w = -1$ and $w = 2$. The global minimum value is -2, taken on at $w = 1$ and $w = -2$.

11. The value at the critical point is $h(0) = 1$, and the value at the other endpoint is e. The global maximum is thus at $z = 1$ and the global minimum is at $z = 0$.

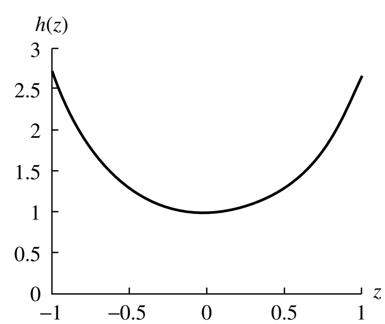
13. $a''(x) = \frac{-2}{(1+x)^3}$, negative for $0 \le x \le 1$. The graph is increasing, concave down.



15. $c''(w) = 6w$. Therefore, $c''(1) = 6$ and $c''(-1) = -6$. This is consistent with the fact that this function has a minimum at 1 and a maximum at -1.



17. $h''(z) = (4z^2 + 2)e^{z^2}$, which is always positive. The critical point at $z = 0$ must be a minimum.



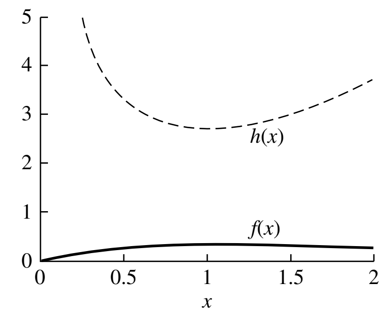
19. a. $h(x) = \frac{1}{f(x)}$, so $h'(x) = \frac{-f'(x)}{f(x)^2}$ by the quotient rule. If $f'(x) = 0$ then $h'(x) = 0$.

b.
$$h''(x) = -\frac{f(x)^2 f''(x) - 2f'(x)^2 f(x)}{f(x)^4}$$

by the quotient rule. If $f'(x) = 0$ then $h''(x) = \frac{-f''(x)}{f(x)^2}$, which has the sign opposite that of the original.

c. If $f(x)$ has a local maximum, $h(x)$ has a local minimum. This makes sense; a large value of $f(x)$ means a small value of $h(x)$, and vice versa.

d. The function is $h(x) = \frac{e^x}{x}$. The derivative is $h'(x) = \frac{e^x(x-1)}{x^2}$, which has a critical point at $x = 1$. The second derivative is $h''(x) = \frac{e^x(x^2 - 2x + 2)}{x^3}$, which is positive at $x = 1$. This is indeed a minimum.



21. a. $h(x) = \ln(f(x))$, so $h'(x) = \frac{f'(x)}{f(x)}$ by the chain rule. If $f'(x) = 0$ then $h'(x) = 0$.

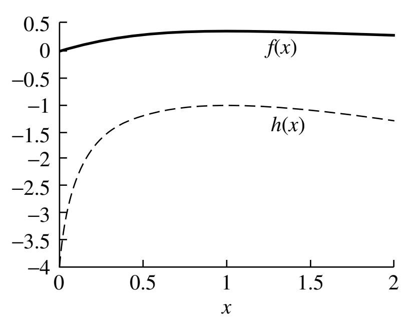
b.

$$h''(x) = \frac{f(x)f''(x) - f'(x)^2}{f(x)^2}$$

by the quotient rule. If $f'(x) = 0$ then $h''(x) = \frac{f''(x)}{f(x)}$, which has the same sign as $f''(x)$.

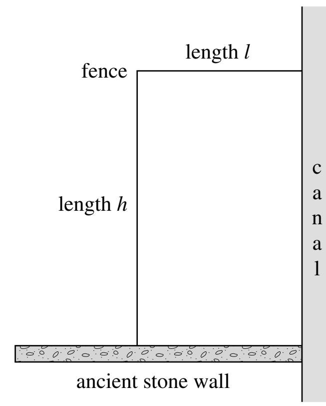
c. If $f(x)$ has a local maximum, $h(x)$ does also. This makes sense; a large value of $f(x)$ means a large value of $h(x)$, and vice versa.

d. The function is $h(x) = \ln(xe^{-x}) = \ln(x) - x$. The derivative is $h'(x) = \frac{1}{x} - 1$, which has a critical point at $x = 1$. The second derivative is $h''(x) = \frac{-1}{x^2}$, which is negative at $x = 1$. This is indeed a maximum.



23. We find that $C'(t) = 3t^2 - 60t$, with critical points at $t = 0$ and $t = 20$. Evaluating at the critical points and endpoints, we find $C(0) = 6000$, $C(20) = 2000$, and $C(25) = 2875$, giving a maximum at $t = 0$ and a minimum at $t = 20$.

25. Consider the following diagram.

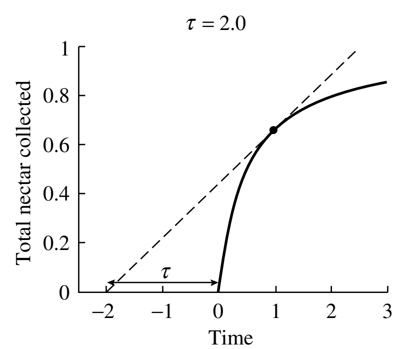


The total length of the fence is $l + h = 1000$, and the area enclosed is lh. We can solve for h as $h = 1000 - l$, so the area is $A(l) = l(1000 - l)$. Then $A'(l) = 1000 - 2l$, which has a critical point at $l = 500$ (meaning that $h = 500$). This is a global maximum because the area at the endpoints $l = 0$ and $l = 500$ is 0. The maximum area is then $500 \cdot 500 = 25{,}000$ m^2.

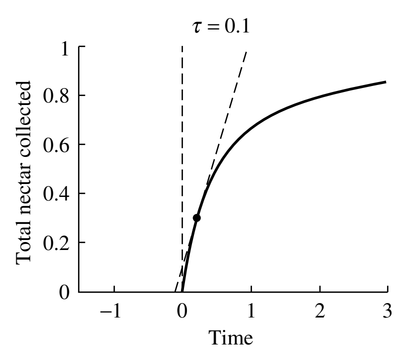
27. Following the steps in Example 3.3.12 gives an optimum of $t = 1.0$. At this point, the derivative is

$$F'(1.0) = \frac{0.5}{(1.0 + 0.5)^2} \approx 0.222$$

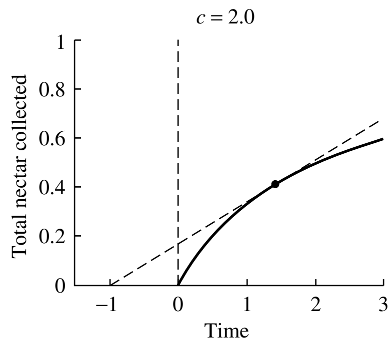
The tangent line is $\hat{F}(t) = F(1) + F'(1)(t-1) = 0.667 + 0.222(t-1)$. We can check directly that $\hat{F}(-2.0) = 0$.



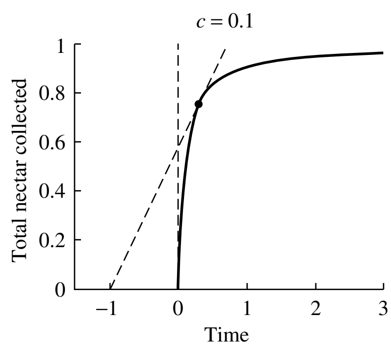
29. $t = \sqrt{0.05} \approx 0.223$. The tangent line is $\hat{F}(t) \approx 0.309 + 0.955(t - 0.223)$. It is true that $\hat{F}(-0.1) = 0$.



31. $t = \sqrt{2}. \approx 1.414$.



33. $t = \sqrt{0.1} \approx 0.316$.



35. The maximum occurs where

$$\frac{0.5}{(t+0.5)^2} = \frac{t}{(t+0.5)(t+\tau)}$$

(following the steps in Example 3.3.12). When $t = 1.0$, we find that

$$\frac{2}{9} = \frac{1.0}{1.5(1.0+\tau)}$$

$$1.5(1.0+\tau) = 4.5$$
$$1.0+\tau = 3.0$$
$$\tau = 2.0$$

The travel time must be 2.0 min.

37. $\tau = 32.0$.

39. $R(t) = \dfrac{t}{(1+t)^2}$, so

$$R''(t) = \frac{2(t-2)}{(t+1)^4}$$

which is negative at $t = 1$, the optimal solution. Therefore, this is at least a local maximum.

41. The bees are trying to maximize $R(n) = \dfrac{n}{P(n)} = \dfrac{n}{1+n^2}$. Taking the derivative, this has a maximum at $n = 1$.

43. $R(n) = \dfrac{n}{P(n)}$, so $R'(n) = \dfrac{P(n) - nP'(n)}{P(n)^2}$. The critical point is where $P(n) = nP'(n)$ or $P'(n) = \dfrac{P(n)}{n}$. If $P(n) = 1 + n^2$, then $P'(n) = 2n$, so the condition is $2n = \dfrac{1+n^2}{n}$, or $2n^2 = 1 + n^2$, or $n = 1$. If $P(n) = 1 + n$, then $P'(n) = 1$, so the condition is $1 = \dfrac{1+n}{n}$, which has no solution.

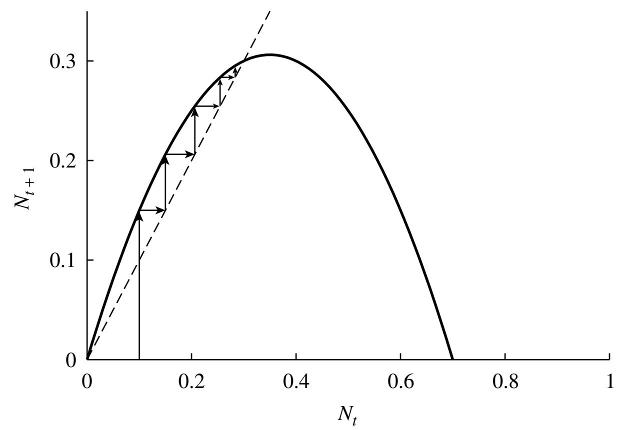
45. a. $N^* = 0$ or $N^* = 1 - \dfrac{1+h}{2.0}$. The largest possible h is 1.

b. $P(h) = h\left(1 - \dfrac{1+h}{2.0}\right)$.

c. $P'(h) = 0$ at $h = 0.5$. This is a maximum because $P''(h) = -1$.

d. $P(0.5) = 0.125$.

47. The derivative of the updating function is $2.5(1 - 2N_t) - h$. The equilibrium is $N^* = 1 - \dfrac{1+h}{2.5}$. Substituting into the derivative gives $h - 0.5$, after some algebra. The equilibrium is stable as long as $h < 1.5$. At $h = 0.75$, the slope is 0.25, indicating stability.

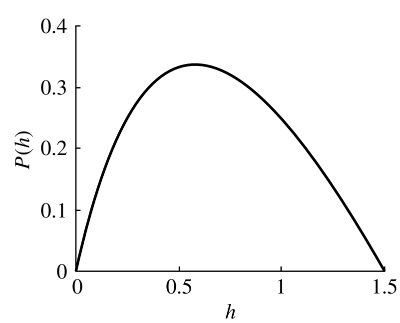


49. a. $N^* = \dfrac{2.5}{1+h} - 1$.

b. $h = 1.5$.

c. $P(h) = hN^*$. The maximum is at $h = \sqrt{2.5} - 1$. This takes on the value of approximately 0.58 for $r = 2.5$.

d. With $r = 2.5$, $P(0.58) \approx 0.338$.



e. The harvest strategy h is lower (0.58 here, whereas it was 0.75 in the text), but the payoff is higher.

51. The equilibrium is $N^* = 1 - \dfrac{1+h}{2.5}$, and the payoff is $P(h) = h\left(1 - \dfrac{1+h}{2.5}\right) - 0.1h$, with derivative $P'(h) = 1 - \dfrac{1+2h}{2.5} - 0.1$.

The critical point is $h = 0.625$, the equilibrium population is $N^* = 0.35$, and the payoff is $P(0.625) \approx 0.156$.

53. The equilibrium is $N^* = 1 - \frac{1+h}{2.5}$, and the payoff is $P(h) = h\left(1 - \frac{1+h}{2.5}\right) - 0.5h$, with derivative $P'(h) = 1 - \frac{1+2h}{2.5} - 0.5$. The critical point is $h = 0.125$, the equilibrium population is $N^* = 0.55$, and the payoff is $P(0.125) \approx 0.062$.

Section 3.4, page 286

1. Let $f(x) = e^x + x^2 - 2$. Then $f(0) = -1 < 0$ and $f(1) = e - 1 > 0$. By the Intermediate Value Theorem, there must be a solution in between.

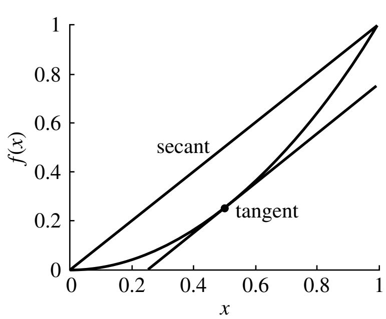
3. To get this into the right form, subtract x from both sides to give the equation $e^x + x^2 - 2 - x = 0$. Let $f(x) = e^x + x^2 - 2 - x$. Then $f(0) = -1 < 0$ and $f(1) = e - 2 > 0$. By the Intermediate Value Theorem, there must be a solution where $f(x) = 0$, or where $e^x + x^2 - 2 - x = 0$. This point must also solve the original equation.

5. Let $f(x) = xe^{-3(x-1)} - 2$. Then $f(0) = -2 < 0$ and $f(1) = -1 < 0$. The Intermediate Value Theorem tells us nothing. However, $f(1/2) = 0.24 > 0$. The Intermediate Value Theorem guarantees solutions between 0 and 1/2 and also between 1/2 and 1.

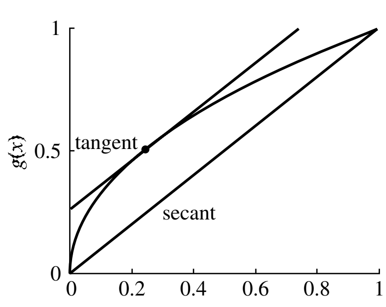
7. $f(0) = f(1) = 0$, and $f(x) > 0$ for $0 \le x \le 1$. Therefore, there must be a positive maximum in that range.

9. $f(0) = f(1) = -1$. There must be a maximum in between. Also, $f(0.5) = 0.875$, which is positive, so there must be a positive maximum.

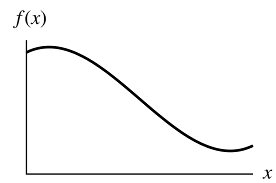
11. $f'(x) = 2x$. The slope of the secant is $\frac{f(1) - f(0)}{1 - 0} = 1$. $f'(x) = 1$ at $x = 0.5$.



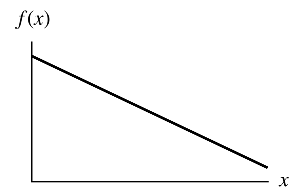
13. $g'(x) = \frac{1}{2\sqrt{x}}$. The slope of the secant is 1, and $g'(x) = 1$ at $x = 1/4$.



15.



17.



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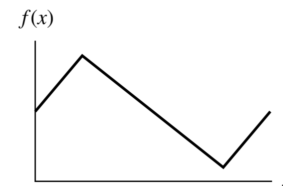


21. The function never takes on any values other than 0 and 1, so it can never equal 1/2. Also, the slope of the secant connecting $x = -1$ and $x = 1$ is 1/2, but the tangent at every point (except at the point of discontinuity $x = 0$) has slope 0.

23. With these values, $f(b) = f(2) = 4$ and $f(a) = f(1) = 1$, so

$$g(x) = x^2 - 3(x - 1)$$

Then $g(1) = g(2) = 1$. Therefore, there must be some value c between $x = 1$ and $x = 2$ where $g'(c) = 0$. But $g'(x) = 2x - 3 = f'(x) - 3$. The point where $g'(x) = 0$ is then a point where $f'(x) = 3$. This is the point guaranteed by the Mean Value Theorem, because the slope of the secant connecting $x = 1$ and $x = 2$ is 3.

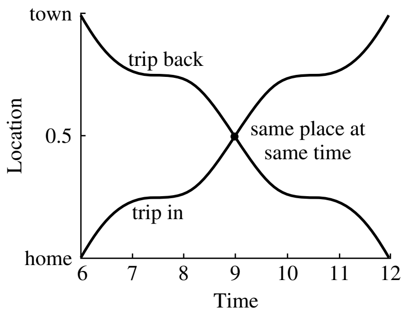
25. The price of gasoline does not change continuously and therefore need not take on all intermediate values.

27. The Intermediate Value Theorem guarantees this crossing. It is possible that it crosses the larger value, but it need not.

29. $x_{t+1} > x_t$ when $x_t = 0$, and $x_{t+1} < x_t$ when $x_t = \pi/2$. Because this discrete-time dynamical system is continuous, there must be a crossing in between.

31. $c_{t+1} > c_t$ when $c_t = 0$, and $c_{t+1} < c_t$ when $c_t = \gamma$. Because this discrete-time dynamical system is continuous, there must be a crossing in between.

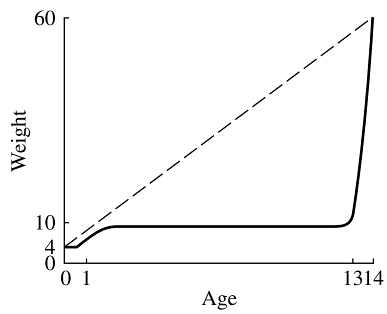
33. The times on her watch during the two trips must look something like the following.



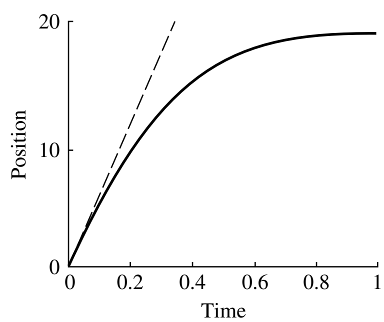
The difference in times on her watch is positive at $t = 6$ and negative at $t = 12$; it must therefore be 0 at some time in between.

35. The Intermediate Value Theorem.

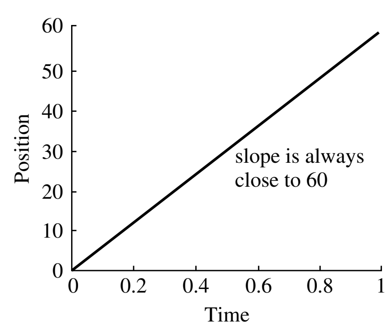
37.



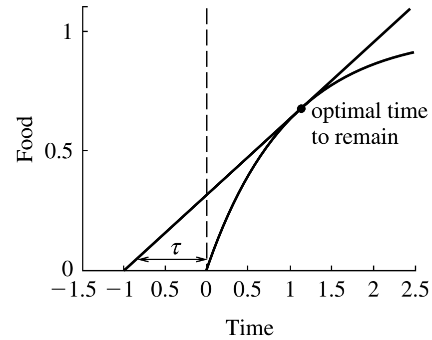
39. The Mean Value Theorem guarantees that the speed at some instant is equal to the average speed of 20 miles per hour. The Intermediate Value Theorem guarantees that the speed will hit *every* value between 0 and 60.



41. The Mean Value Theorem guarantees that the speed at some instant will be equal to the average speed of 60 mph. The Intermediate Value Theorem guarantees that the speed must hit *every* value between the minimum and maximum speed, but we do not know what those speeds are.



43.



Section 3.5, page 297

1. 0.

3. 0.

5. ∞.

7. 0.

9. e^{2x} approaches infinity faster because an exponential function always beats a power function. For x^2, the values are 1, 100, and 10,000. For e^{2x}, the values are 7.3, 4.85×10^8 and 7.22×10^{86}, which are always larger.

11. $0.1x^{10}$ approaches infinity faster because the power is larger. For $x^{3.5}$, the values are 1, 3162, and 10^7. For $0.1x^{10}$, the values are 0.1, 10^9, and 10^{19}. By the time $x = 10$, $0.1x^{10}$ is larger.

13. $0.1x^{0.5}$ approaches infinity faster because a power function beats the natural log. For $0.1x^{0.5}$, the values are 0.1, 0.316, and 1.000. For $30\ln(x)$, the values are 0, 69.1, and 138.1. These are much larger. In fact, the two functions don't get into the right order until x is around 3×10^7.

15. e^{-2x} approaches 0 faster because exponentials are faster than power functions. For e^{-2x}, the values are 0.135, 2.1×10^{-9} and 1.38×10^{-87}. For x^{-10}, the values are 1, 10^{-10}, and 10^{-20}. The two functions don't get into the right order until $x = 100$.

17. $x^{-3.5}$ approaches 0 faster because the power is more negative. For $1000/x$, the values are 1000, 100, and 10. For $x^{-3.5}$, the values are 1.00, 0.0032, and 1.0×10^{-7}. The two functions are always in the right order.

19. x^{-2} approaches 0 faster because power functions are faster than natural logs. For x^{-2}, the values are 1, 0.01, and 0.0001. For $30/\ln(x)$, the values are undefined (division by 0), 13.03, and 6.51. The two functions are always in the right order.

21. Approaches 0 because the denominator is an exponential function, which grows more quickly than the quadratic in the numerator.

23. Approaches 0 because the denominator is an exponential function with a larger coefficient.

25. The derivative is 5, a constant, as is consistent with linearity.

27. The derivative is $\alpha'(c) = \dfrac{10c}{(1+c^2)^2}$, so $\alpha'(0) = 0$ and the limit as c approaches infinity is 0. This is consistent with a graph that starts out flat, then increases, and eventually flattens out again.

29. The derivative is $\alpha'(c) = \frac{5(1-c^2)}{(1+c^2)^2}$, so $\alpha'(0) = 5$. The limit as c approaches infinity is 0, but the derivative is negative. This is consistent with a graph that starts out increasing but then decreases to 0.

31. This population increases to infinity because $r > 1$. $b_t = 10^8 1.1^t > 10^{10}$ if

$$\begin{aligned} 1.1^t &> 10^2 \\ e^{\ln(1.1)t} &> 10^2 \\ \ln(1.1)t &> \ln(10^2) \\ t &> \ln(10^2)/\ln(1.1) \approx 48.3 \end{aligned}$$

The population exceeds the threshold after about 49 generations.

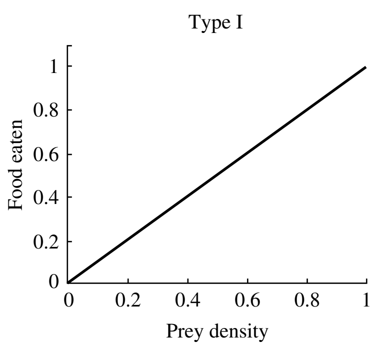
33. This population decreases to 0 because $r < 1$. Solving $b_t = 10^3$ gives that it takes 16.60 generations to reach 10^3.

35. $l_t = 2t$ and $r_t = 2^{t+1}$. The ratios are 0.5, 0.156, 0.0098, and 0.000019.

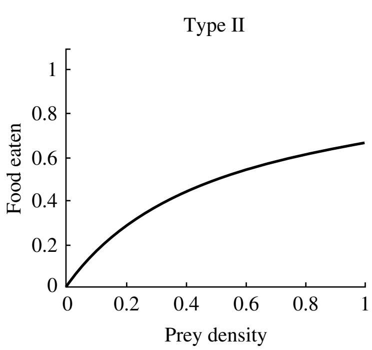
37. The equilibrium is $M^* = 2.0$.

39. We can solve $M_t = 2.02$, or $0.5^t \cdot 3.0 = 0.02$, or $0.5^t = 0.0067$, or $t = 7.2$.

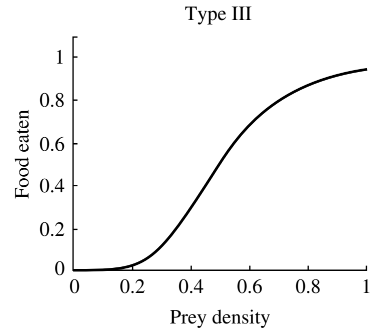
41. They look like linear, saturated, and saturated with threshold.



a



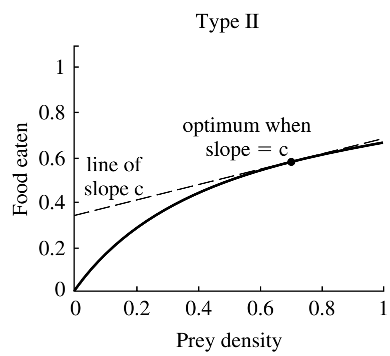
b



c

In each case, the predator is happiest with an infinite supply of prey.

43.



45. The optimal prey density is 0 if $c > 1$ and infinity $c < 1$.

47. The lung becomes less efficient at absorbing chemical as the concentration increases, but it is always able to take up a little more.

49. When chemical concentration is too high, the lung is poisoned.

Section 3.6, page 309

1. $f_0(x) = 1$, $f_\infty(x) = x$.

3. $h_0(z) = e^z$, $h_\infty(z) = e^z$.

5. $m_0(a) = \frac{1}{a}$, $m_\infty(a) = 30a^2$.

7. The exponential function e^{2x} approaches infinity faster. By L'Hôpital's rule,

$$\begin{aligned} \lim_{x\to\infty} \frac{e^{2x}}{x^2} &= \lim_{x\to\infty} \frac{2e^{2x}}{2x} \\ &= \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty \end{aligned}$$

9. The power function $0.1x^{0.5}$ approaches infinity faster. By L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{0.1x^{0.5}}{30\ln(x)} = \lim_{x\to\infty} \frac{0.05x^{-0.5}}{30x^{-1}} = \lim_{x\to\infty} 0.0017x^{0.5} = \infty$$

11. The exponential function e^{-2x} approaches 0 faster. L'Hôpital's rule doesn't really make things simpler directly, but

$$\lim_{x\to\infty} \frac{e^{-2x}}{x^{-2}} = \lim_{x\to\infty} \frac{x^2}{e^{2x}} = \lim_{x\to\infty} \frac{2x}{2e^{2x}} = \lim_{x\to\infty} \frac{2}{4e^{2x}} = 0$$

13. I would guess that the power function approaches infinity faster.

$$\lim_{x\to 0}\frac{x^{-1}}{-\ln(x)}=\lim_{x\to 0}\frac{-x^{-2}}{-x^{-1}}=\lim_{x\to 0}x^{-1}=\infty$$

Therefore, $\lim_{x\to 0}\dfrac{1}{x\ln(x)}=\infty$, and $\lim_{x\to 0}x\ln(x)=0$.

15. The power function with the larger power, x^3, approaches 0 more quickly.

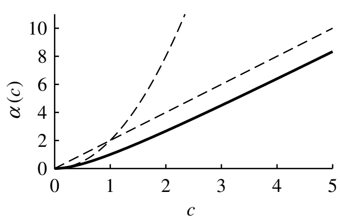
$$\lim_{x\to 0}\frac{x^3}{x^2}=\lim_{x\to 0}\frac{3x^2}{2x}=\lim_{x\to 0}\frac{6x}{2}=0$$

17. The numerator has only one term, so the leading behavior is $2c^2$ at both 0 and ∞. The denominator has leading behavior 1 for c near 0 and has leading behavior c for c large. Therefore,

$$\alpha_0(c)=\frac{2c^2}{1}=2c^2$$
$$\alpha_\infty(c)=\frac{2c^2}{c}=2c$$
$$\lim_{c\to 0}\alpha(c)=0$$
$$\lim_{c\to\infty}\alpha(c)=\infty$$

L'Hôpital's rule is not appropriate at $c=0$, because the denominator approaches 1. This limit can be found by substituting. As $c\to\infty$, both the numerator and the denominator approach infinity, so we can use L'Hôpital's rule to check

$$\lim_{c\to\infty}\frac{2c^2}{1+c}=\lim_{c\to\infty}\frac{4c}{1}=\infty$$

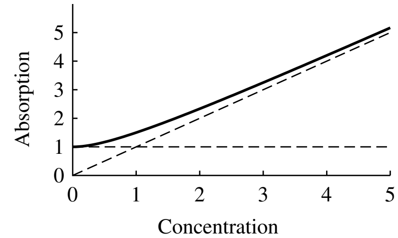


19. The numerator has leading behavior 1 near 0 and leading behavior c^2 for c large. The denominator has leading behavior 1 for c near 0 and leading behavior c for c large. Therefore,

$$\alpha_0(c)=\frac{1}{1}=1$$
$$\alpha_\infty(c)=\frac{c^2}{c}=c$$
$$\lim_{c\to 0}\alpha(c)=1$$
$$\lim_{c\to\infty}\alpha(c)=\infty$$

L'Hôpital's rule is not appropriate at $c=0$, because both the numerator and the denominator approach 1. This limit can be found by substituting. As $c\to\infty$, both the numerator and the denominator approach infinity, so we can use L'Hôpital's rule to check

$$\lim_{c\to\infty}\frac{1+c+c^2}{1+c}=\lim_{c\to\infty}\frac{1+2c}{1}=\infty$$

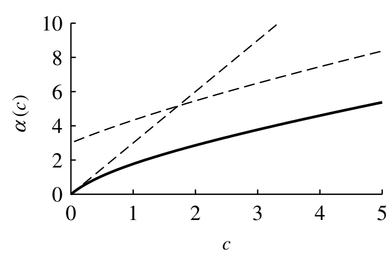


21. The numerator has only one term, so the leading behavior is $3c$ at both 0 and ∞. The denominator has leading behavior 1 for c near 0 and leading behavior $\ln(1+c)$ for c large. Therefore,

$$\alpha_0(c)=\frac{3c}{1}=3c$$
$$\alpha_\infty(c)=\frac{3c}{\ln(1+c)}$$
$$\lim_{c\to 0}\alpha(c)=0$$
$$\lim_{c\to\infty}\alpha(c)=\infty$$

L'Hôpital's rule is not appropriate at $c=0$, because the denominator approaches 1. This limit can be found by substituting. As $c\to\infty$, both the numerator and the denominator approach infinity, so we can use L'Hôpital's rule to check

$$\lim_{c\to\infty}\frac{3c}{1+\ln(1+c)}=\lim_{c\to\infty}\frac{3}{\frac{1}{1+c}}=\lim_{c\to\infty}\frac{3(1+c)}{1}=\infty$$



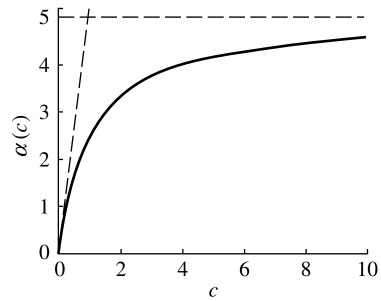
23. The tangent line to $2x+x^2$ near $x=0$ is $2x$. The tangent line to $3x+2x^2$ near $x=0$ is $3x$. For small x, $f(x)\approx\dfrac{2x}{3x}=2/3$. With L'Hôpital's rule,

$$\lim_{x\to 0}\frac{2x+x^2}{3x+2x^2}=\lim_{x\to 0}\frac{2+2x}{3+4x}=2/3$$

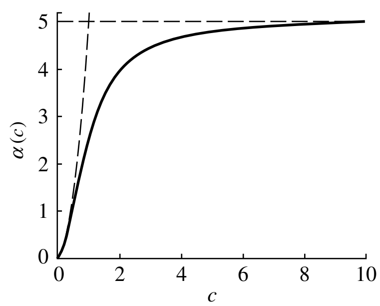
25. The tangent line to $\ln(x)$ near $x=1$ is $x-1$. The tangent line to x^2-1 near $x=1$ is $2(x-1)$. For small x, $f(x)\approx\dfrac{x-1}{2(x-1)}=1/2$. With L'Hôpital's rule,

$$\lim_{x\to 1}\frac{\ln(x)}{x^2-1}=\lim_{x\to 1}\frac{1/x}{2x}=1/2$$

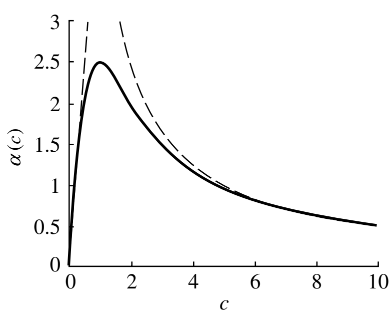
27. This function acts like $5c$ for small c and like 5 for large c.



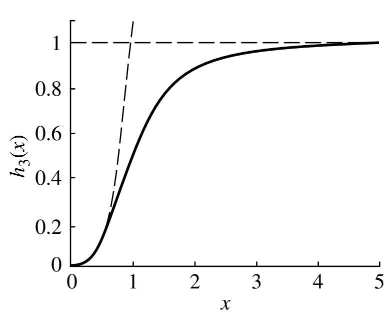
29. This function acts like $5c^2$ for small c and like 5 for large c.



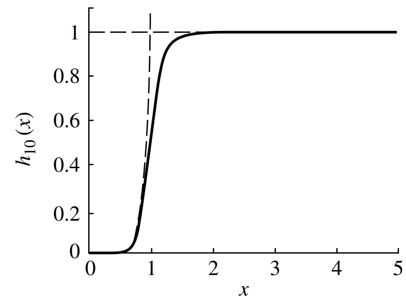
31. This function acts like $5c$ for small c and decreases to 0 like $5/c$ for large c.



33. $h_3(x)_0 = x^3$, $h_3(x)_\infty = 1$.



35. $h_{10}(x)_0 = x^{10}$, $h_{10}(x)_\infty = 1$.



37. a. $a_t = 10^4 \cdot 2.0^t$, $b_t = 10^6 \cdot 1.5^t$.

b. $p_t = \dfrac{10^4 \cdot 2.0^t}{10^4 \cdot 2.0^t + 10^6 \cdot 1.5^t}$.

c. In exponential notation, we can rewrite the denominator as $10^4 e^{\ln(2.0)t} + 10^6 e^{\ln(1.5)t}$, with leading behavior $10^4 e^{\ln(2.0)t}$, because the parameter in the exponent is larger. Therefore, $\lim_{t\to 0} p_t = 1$.

d. $p_0 = 0.01$, $p_{10} \approx 0.15$, $p_{20} \approx 0.76$, $p_{50} \approx 0.9999$. This is mighty close to the limit.

39. a. $a_t = 10^4 \cdot 0.8^t$, $b_t = 10^5 \cdot 1.2^t$.

b. $p_t = \dfrac{10^4 \cdot 0.8^t}{10^4 \cdot 0.8^t + 10^5 \cdot 1.2^t}$.

c. In exponential notation, the denominator is $10^4 e^{\ln(0.8)t} + 10^5 e^{\ln(1.2)t} = 10^4 e^{-0.223t} + 10^5 e^{0.182t}$. The leading behavior is $10^4 e^{0.182t}$ because this term grows and the other term shrinks. Therefore, $\lim_{t\to 0} p_t = 0$.

d. $p_0 = 0.09$, $p_{10} \approx 0.0017$, $p_{20} \approx 0.00003$, $p_{50} \approx 1.5 \times 10^{-10}$. This is extremely close to the limit.

41. $r(c) = c$, the derivative is $\alpha'(c) = \dfrac{Ak}{(k+c)^2} > 0$. By L'Hôpital's rule,

$$\lim_{c\to\infty} A\frac{c}{k+c} = \lim_{c\to\infty} A\frac{1}{1} = A$$

Near $c = 0$, the leading behavior of the denominator is k, so $\alpha(c) \approx \dfrac{c}{k}$. For large c, the leading behavior of the denominator is c, so $\alpha(c) \approx \alpha$.

43. With $r(c) = c^n$, the derivative is $\alpha'(c) = \dfrac{nAkc^{n-1}}{(k+c^n)^2} > 0$. By L'Hôpital's rule,

$$\lim_{c\to\infty} A\frac{c^n}{k+c^2} = \lim_{c\to\infty} A\frac{nc^{n-1}}{nc^{n-1}} = A.$$

Near $c = 0$, the leading behavior of the denominator is k, so $\alpha(c) \approx \dfrac{c^n}{k}$. For large c, the leading behavior of the denominator is c^n, so $\alpha(c) \approx \alpha$.

Section 3.7, page 319

1. Let $f(x) = x^3$ with base point $a = 2.0$. To find the tangent line approximation, evaluate $f(2) = 8.0$ and $f'(2) = 12.0$. The tangent line is then $\hat{f}(x) = 8.0 + 12.0(x - 2.0)$, and $\hat{f}(2.02) = 8.0 + 12.0(2.02 - 2.0) = 8.0 + 12.0 \cdot 0.02 = 8.24$. To find the secant line, we evaluate $f(3) = 27$, so the secant line has slope 19. Therefore, $f_s(t) = 8.0 + 19.0(x - 2.0)$ and $f_s(2.02) = 8.0 + 19.0 \cdot 0.02 = 8.38$. The exact answer is 8.242408, which is pretty close to the tangent line approximation.

3. Let $f(x) = \sqrt{x}$ with base point 4.0. $f(4) = 2$, $f'(x) = \frac{1}{2}x^{-1/2}$, and $f'(4) = 0.25$. Thus $\hat{f}(x) = 2 + 0.25(x - 4)$ and $\hat{f}(4.01) = 2.0025$. To find the secant line, we evaluate $f(9) = 3$, so the secant line has slope 0.2. Therefore, $f_s(t) = 2.0 + 0.2(x - 4.0)$ and $f_s(4.01) = 2.0 + 0.2 \cdot 0.01 = 2.002$. The exact answer is 2.002498 to six decimal places, quite close to the tangent line approximation.

5. Let $f(x) = \sin(x)$. $f(0) = \sin(0) = 0$. $f'(x) = \cos(x)$ and $f'(0) = 1$. Thus $\hat{f}(x) = 0 + 1(x - 0)$ and $\hat{f}(0.02) = 0.02$. The secant line does not help because there is no easy value of x to evaluate this function. The exact answer is 0.19998 to five decimal places.

7. Let $f(x) = x^3$ with base point $a = 2.0$. $f''(2) = 12.0$, so $P_2(x) = 8.0 + 12.0(x - 2.0) + 6.0(x - 2.0)^2$ and $P_2(2.02) = 8.0 + 12.0(2.02 - 2.0) + 6.0(2.02 - 2.0)^2 = 8.0 + 12.0 \cdot 0.02 + 6.0 \cdot 0.0004 = 8.2424$. The exact answer is 8.242408, which is very close to the quadratic approximation.

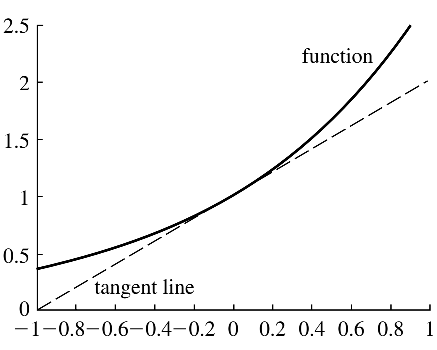
9. Let $f(x) = \sqrt{x}$ with base point 4.0. $f''(4) = -1/32$, so $P_2(x) = 2 + 0.25(x - 4) - (x - 4)^2/64$ and $P_2(4.01) = 2.002498438$. The exact answer is 2.0024984395 to 10 decimal places.

11. Let $f(x) = \sin(x)$ with base point 0.0. $f''(x) = -\sin(x)$ so $f''(0) = 0$. Then $P_2(x) = 0 + 1(x - 0)$, which is identical to the tangent line. The exact answer is 0.19998 to five decimal places.

13. This is a composition of $f(g(x))$ where $g(x) = 1 + 3x$ and $f(g) = g^2$. Near the base point $x = 1$, we have that $f(g(1)) = 16$, and $(f \circ g)'(x) = 6(1 + 3x)$, so $(f \circ g)'(1) = 24$. The tangent line approximation is $16 + 24(x - 1)$ which has value 16.24 at $x = 1.01$. In steps, the tangent line to g at $x = 1$ is $\hat{g}(x) = 4 + 3(x - 1)$, so $\hat{g}(1.01) = 4.03$. ($\hat{g} = g$ because g is linear). We then evaluate f near 4, finding $f(4) = 16$ and $f'(4) = 8$. Then $\hat{f}(x) = 16 + 8(x - 4)$ and $\hat{f}(4.03) = 16.24$, exactly as before.

15. Both methods give 1.02

17. $e^{0.1} \approx 1.105 > 1.1$. $e^{-0.1} \approx 0.905 > 0.9$. The estimates are low because the graph of e^x is concave up and lies above the tangent.



19. $1.1^2 = 1.21 > 1.2$. $0.9^2 = 0.81 > 0.8$. The estimates are low because the graph of x^2 is concave up and lies above the tangent.

21. $P_3(x) = 1 + 3x + 4x^2 + x^3$.

23. $f(1) = 8$, $f'(1) = 11$, $f''(1) = 8$, $f'''(x) = 0$, so $P_3(x) = 8 + 11(x - 1) + 4(x - 1)^2$.

25. $h(1) = 0$, $h'(1) = 1$, $h''(1) = -1$, and $h'''(1) = 2$. Therefore, $P_3(x) = x - 1 - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$.

27. The Taylor polynomials are

$$P_1(x) = 1 + x$$
$$P_2(x) = 1 + x + x^2$$
$$P_3(x) = 1 + x + x^2 + x^3$$
$$P_4(x) = 1 + x + x^2 + x^3 + x^4$$

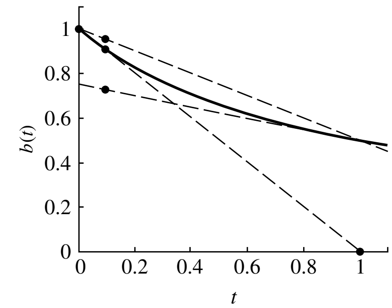
and so forth. Because the Taylor polynomials approximate the function f, the sum of the series is $f\left(\frac{1}{3}\right) = \frac{3}{2}$. I added up the first five terms $\left(\text{up to} \frac{1}{3^5}\right)$ and got 1.4979.

29. $\alpha(0) = 0$, and $\alpha'(c) = \frac{5}{(1 + c)^2}$, so $\alpha'(0) = 5$. Therefore, the tangent line is $\hat{A}(c) = 5c$, matching the result found with leading behavior.

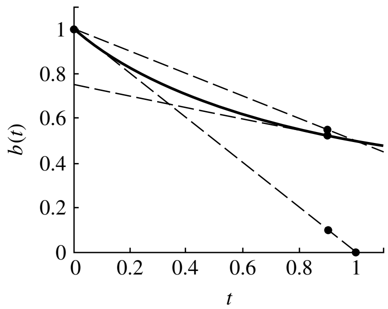
31. $\alpha(0) = 0$, and $\alpha'(c) = \frac{10c}{(1 + c^2)^2}$, so $\alpha'(0) = 0$. Therefore, the tangent line is $\hat{A}(c) = 0$, which does not match the result found with leading behavior. We would need the quadratic approximation to match the leading behavior.

33. $\alpha(0) = 0$, and $\alpha'(c) = \frac{5}{1 + c^2} - \frac{10c^2}{(1 + c^2)^2}$, so $\alpha'(0) = 5$. Therefore, the tangent line is $\hat{A}(c) = 5c$, matching the result found with leading behavior.

35. The tangent at $t = 0$ gives 0.9, the tangent at $t = 1$ gives 0.725, and the secant gives 0.95. The exact answer is 0.9091, closest to the tangent at $t = 0$. The secant is in the right ball park, and the tangent at $t = 1$ is way off.



37. The tangent at $t = 0$ gives 0.1, the tangent at $t = 1$ gives 0.525, and the secant gives 0.55. The exact answer is 0.526, closest to the tangent at $t = 1$. The secant is in the right ball park, and the tangent at $t = 0$ is way off.

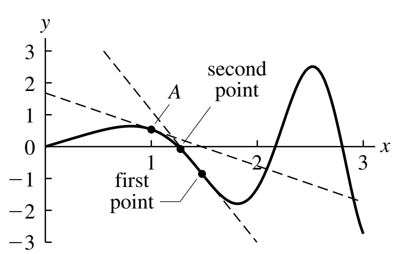


39. Using the base point $a = 1.0$, $M(1) = 1$. $M'(a) = 3a^2$ so $M'(1) = 3$. Then the tangent line is $\hat{M}(a) = 1 + 3(a - 1)$ and $\hat{M}(1.25) = 1 + 3(0.25) = 1.75$. The secant line connecting $a = 1$ and $a = 1.5$ is $M_s(a) = 3.375 + 4.75(a - 1.5)$, and $M_s(1.25) = 3.375 + 4.75(-0.25) = 2.1875$. The exact value is approximately 1.953. Neither of the approximations is very close, but only the secant line would be possible if we did not know the formula.

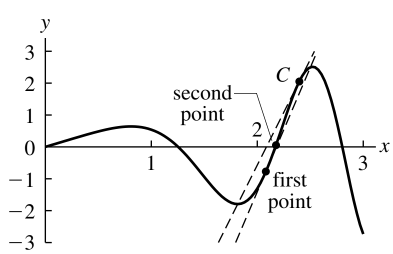
41. First, we can use the equation $T(t) = t + t^2$ to find the tangent line at $t = 0$ as $\hat{T}(t) = t$, and estimate $T(1) = 1$. We can find the tangent line at $t = 2$ as $\hat{T}(t) = 6 + 5(t - 2)$, and again estimate $T(1) = 1$. The secant line between the actual data points has slope $\frac{6.492 - 0.172}{2} = 3.16$, giving the line $T_s(t) = 0.172 + 3.16t$ with the value $T(1) = 3.332$. None of these is very close to the correct answer.

Section 3.8, page 328

1.



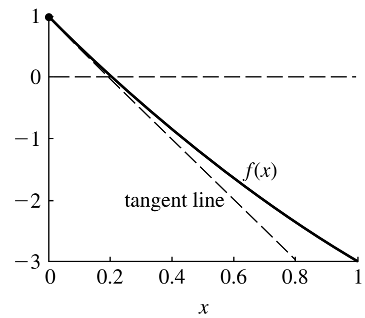
3.



5. Set $f(x) = x^2 - 5x + 1$. Then $f(0) = 1$ and $f(1) = -3$, so by the Intermediate Value Theorem, there must be a solution in between. If we start with a guess of $x = 0$, we find $f'(0) = -5$, so the tangent line is $\hat{f}(x) = 1 - 5x$, which intersects the horizontal axis at $x = 0.2$. The Newton's method discrete-time dynamical system is

$$x_{t+1} = x_t - \frac{f(x_t)}{f'(x_t)} = x_t - \frac{x_t^3 - 5x_t + 1}{2x_t - 5}$$

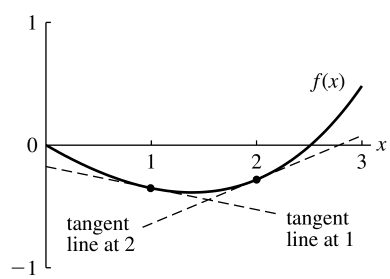
If $x_0 = 0$, then $x_1 = 0.2$, $x_2 \approx 0.2086956522$, and $x_3 \approx$ 0.2087121525. The exact answer is 0.2087121525. . . .



7. Suppose our initial guess is $x_0 = 1$. Because $h'(x) = 0.5e^{x/2} - 1$, $h(1) = -0.351$, and $h'(1) = -0.1756$, the tangent line is $\hat{h}(x) = -0.351 - 0.1756(x - 1)$, which intersects the horizontal axis at $x = -1$. The Newton's method discrete-time dynamical system is

$$x_{t+1} = x_t - \frac{h(x_t)}{h'(x_t)} = x_t - \frac{e^{x_t/2} - x_t - 1}{0.5e^{x_t/2} - 1}$$

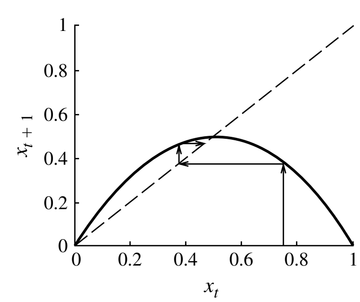
If $x_0 = 1$, then $x_1 = -1.00$, $x_2 = -0.1294668027$, and $x_3 = -0.0037771286$. This seems to be approaching 0. If we start from $x_0 = 2$, we get $x_1 = 2.7844$, $x_2 = 2.5479$, and $x_3 = 2.51355$. After many steps, the answer converges to 2.512862414.



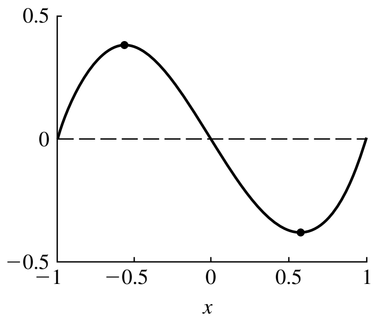
9. It is true that $x = e^x - 2$ if $e^x = x + 2$. But starting from $x_0 = 1$, solutions are $x_1 \approx 0.718$, $x_2 \approx 0.051$, and $x_3 \approx -0.947$. After a while, it seems to converge to another equilibrium at –1.84.

11. It is true that $e^{x/2} = x + 1$ if $h(x) = 0$. Starting from a guess of $x_0 = 2$, we find $x_1 = 1.718$, $x_2 = 1.361$, and $x_3 = 0.975$. This seems to be going very slowly to $x = 0$.

13. It should converge most rapidly when the equilibrium is superstable, or the slope is 0. The slope at the equilibrium is $2 - r$, so the most rapid convergence should be when $r = 2$. Starting from $x_0 = 0.75$, we get $x_1 = 0.375$, $x_2 \approx 0.46875$, $x_3 \approx 0.4980468750$, and $x_4 \approx 0.4999923706$.



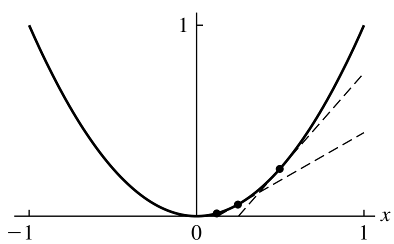
15. The method fails if we start at a critical point of the function, and these points occur where $x = \pm\frac{\sqrt{3}}{3} \approx \pm 0.577$. All values below the lower critical point converge to the negative solution.



17. The Newton's method discrete-time dynamical system is

$$x_{t+1} = x_t - \frac{x_t^2}{2x_t} = \frac{x_t}{2}$$

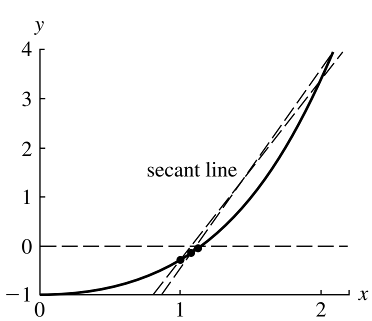
This converges to 0 rather slowly, because the derivative of the function x^2 is 0 at the solution.



19. $f'(x) \approx e^{x+1} - (x+1) - 2 - (e^x - x - 2) = e^{x+1} - 1 - e^x$. The approximate Newton's method discrete-time dynamical system is

$$x_{t+1} = x_t - \frac{e^{x_t} - x_t - 2}{e^{x_t+1} - 1 - e^{x_t}}$$

After five steps, it has gotten only to 1.208, not too close to the exact answer of 1.14619, to 5 decimal places.



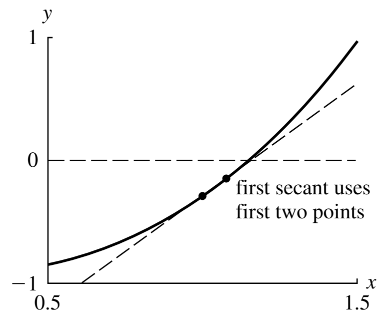
21. The discrete-time dynamical system is

$$x_{t+1} = x_t - \frac{f(x_t)(x_t - x_{t-1})}{f(x_t) - f(x_{t-1})}$$

which depends on both x_t and x_{t-1}, unlike an ordinary discrete-time dynamical system. Starting from $x_0 = 1.0$ and

$$x_1 = x_0 - \frac{e^{x_0} - x_0 - 2}{e^{x_0+1} - 1 - e^{x_0}}$$

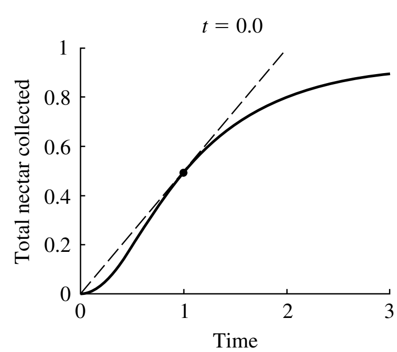
from Exercise 19, we find $x_1 \approx 1.076746253$, $x_2 \approx 1.154339800$, $x_3 \approx 1.145768210$, $x_4 \approx 1.146190691$, and $x_5 \approx 1.146193221$, which is correct to eight decimal places. This method is better because it uses a more accurate version of the secant line approximation.



23. This function differs in that it is concave up for small t. The nectar comes out slowly when the bee first arrives. The derivative is $F'(t) = \frac{2t}{(1+t^2)^2}$, so the Marginal Value Theorem equation is

$$\begin{aligned} F'(t) &= \frac{F(t)}{t} \\ \frac{2t}{(1+t^2)^2} &= \frac{t}{1+t^2} \\ \frac{2}{1+t^2} &= 1 \\ 1+t^2 &= 2 \\ t^2 &= 1 \\ t &= 1 \end{aligned}$$

It turned out that this could be solved algebraically.



25. $N^* = \ln\left(\frac{2.5}{1+h}\right)$. Therefore, $P(h) = hN^* = h\ln\left(\frac{2.5}{1+h}\right)$. Then

$$P'(h) = \ln\left(\frac{2.5}{1+h}\right) - \frac{h}{1+h} = 0$$

We can use this function as f in Newton's method. I used an initial guess of $h_0 = 0.75$ because this is the solution we found when the fish followed the logistic model. After three steps, we see that the solution is $h \approx 0.6724$.

27. Let G be the updating function. Then $G(0) = 1$ and $G(5) \approx 2.516$. There must be an equilibrium in between. Starting from $M_0 = 2$, get to 1.726 after about six steps. Solving the equation $0.5e^{-0.1x}x + 1.0 - x = 0$ with Newton's method, it reaches 1.726 after two steps.

29. We need to solve the equation $100e^{0.1t} = 400 + 100t$, or $100e^{0.1t} - 400 - 100t = 0$. The Newton's method discrete-time dynamical system is

$$x_{t+1} = x_t - \frac{100e^{0.1x_t} - 400 - 100x_t}{10e^{0.1x_t} - 100}$$

If food resources were constant at the initial value of 400, the population would run out of food when $b(t) = 400$, or at time $10\ln(4) \approx 13.8$. When we use an initial guess of $x_0 = 13.8$, the solution shoots off to a negative value. Using an initial guess of $x_0 = 30$ instead, we find that the solution as $t = 37.18$ after three steps.

31. We will start with the guess $c_0 = 5.0$, because the equilibrium is between 0 and γ. We need to solve the equation $c^* = 0.75e^{-c^*}c^* + 1.25$ or $f(c) = 0$ where $f(c) = c - 0.75e^{-c^*}c^* - 1.25$. Then $f'(c) = 1 + 0.75e^{-c}c - 0.75e^{-c}$, and the Newton's method discrete-time dynamical system is

$$c_{t+1} = c_t - \frac{c_t - 0.75e^{-c_t}c_t - 1.25}{1 + 0.75e^{-c_t}c_t - 0.75e^{-c_t}}$$

With $c_0 = 5.0$, $c_1 \approx 1.349066687$, $c_2 \approx 1.502145217$, and $c_3 \approx 1.500942584$.

Section 3.9, page 334

1. The absorption rate is

$$R(T) = \frac{2.5}{1.5 - 0.5T}$$

The derivative is

$$R'(T) = \frac{1.25}{(1.5 - 0.5T)^2}$$

which is always positive. Therefore, the maximum must be at $T = 1$, where $R(1) = 2.5$.

3. The absorption rate is

$$R(T) = \frac{5.0\alpha}{1.0 + \alpha - \alpha T}$$

The derivative of $R(T)$ is

$$R'(T) = \frac{5.0\alpha^2}{(1.0 + \alpha - \alpha T)^2}$$

which is always positive. Therefore, the maximum must be at $T = 1$, where

$$R(1) = 5.0\alpha$$

This is an increasing function of α because α translates breath duration into absorption.

5. Equation 3.9.8 says that

$$c^* = \frac{\gamma r T}{1 - (1 - rT)[1 - \alpha(1 - e^{-kT})]}$$

when the discrete-time dynamical system is

$$c_{t+1} = (1 - rT)[c_t - \alpha c_t(1 - e^{-kT})] + rT\gamma.$$

The equilibrium is

$$\begin{aligned} c^* &= (1 - rT) && \text{equation for} \\ &\quad \times [c^* - \alpha c^*(1 - e^{-kT})] + r\gamma T && \text{equilibrium} \\ r\gamma T &= c^* - (1 - rT) && \text{move all the } c^*\text{'s} \\ &\quad \times [c^* - \alpha c^*(1 - e^{-kT})] && \text{to one side} \\ r\gamma T &= c^*\{1 - (1 - rT) && \\ &\quad \times [1 - \alpha(1 - e^{-kT})]\} && \text{factor out a } c^* \\ c^* &= \frac{\gamma r T}{1 - (1 - rT)[1 - \alpha(1 - e^{-kT})]} && \text{solve for } c^* \end{aligned}$$

7. The rate of absorption is

$$R(T) = \frac{1.25\dfrac{T^2}{0.1 + T^2}}{1 - (1 - 0.5T)\left(1 - 0.5\dfrac{T^2}{0.1 + T^2}\right)}$$

With much patience, it is possible to take the derivative to find the answer.

9. The optimal T does not depend on r at all. I have no idea why.

11. The equilibrium is

$$c^* = \frac{\gamma r T}{1 - (1 - rT)[1 - A(T)]}$$

13.

$$c^* = \frac{\gamma r T}{1 - (1 - rT)(1 - \alpha T)}$$

$$\frac{c^* A(T)}{T} = \frac{\alpha\gamma r T}{1 - (1 - rT)(1 - \alpha T)}$$

15.

$$c^* = \frac{\gamma r T}{1 - (1 - rT)\left(1 - \alpha\dfrac{T^2}{k + T^2}\right)}$$

$$\frac{c^* A(T)}{T} = \frac{\alpha\gamma r\dfrac{T^2}{k + T^2}}{1 - (1 - rT)\left(1 - \alpha\dfrac{T^2}{k + T^2}\right)}$$

17. We cannot substitute $T = 0$ because

$$\frac{c^* A(T)}{T} = \frac{\alpha\gamma r T}{1 - (1 - rT)(1 - \alpha T)}$$

is an indeterminate form. Taking the derivative of top and bottom with respect to T gives

$$\lim_{T\to 0}\frac{c^*A(T)}{T}=\lim_{T\to 0}\frac{\alpha\gamma r}{r+\alpha-r\alpha T}=\frac{\alpha\gamma r}{r+\alpha}.$$

This is positive, because the lung still absorbs some chemical, even when breathing is extremely fast.

19. This is also an indeterminate form.

$$\lim_{T\to 0}\frac{c^*A(T)}{T}=\lim_{T\to 0}\frac{\alpha\gamma r T^2}{1-(1-rT)(1-\alpha T^2)}$$
$$=\lim_{T\to 0}\frac{2\alpha\gamma r T}{2\alpha T(1-rT)+r(1-\alpha T^2)}$$
$$=\frac{0}{0+r}=0$$

There is no absorption when the animal pants. This makes sense because this lung absorbs very slowly when T is small.

21. The equilibrium value is

$$c^*=\frac{r\gamma(k+T)}{\alpha+rk+rT-rTa}$$

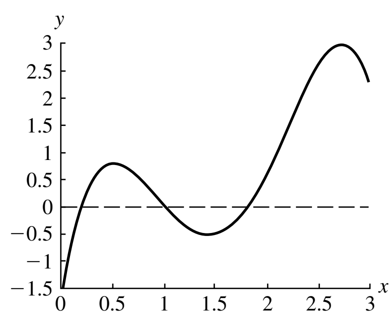
The equilibrium absorption rate is then

$$R(T)=\frac{\alpha r\gamma}{\alpha+rk+rT-rT\alpha}$$

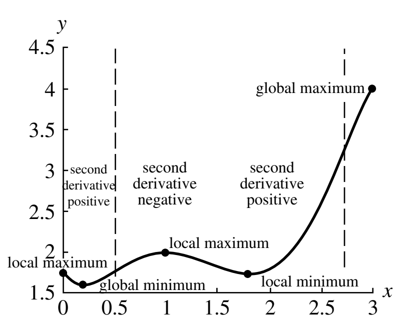
As long as $\alpha<1$, this is a decreasing function of T, meaning that the organism would do best to pant no matter what the parameter values.

Supplementary Problems, page 335

1. a.



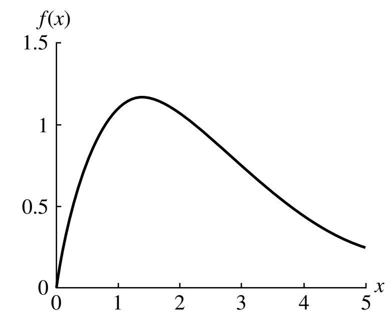
d.



3. Let $f(x)=1/(3+x^2)$. Then $f'(x)=-2x/(3+x^2)^2$. Substituting $x=1$, we find $f(1)=1/4$ and $f'(1)=-1/8$, so $\hat{f}(x)=1/4-1/8(x-1)$, and $\hat{f}(1.01)=1/4-1/8(1.01-1)=0.24875$.

5. $\hat{f}(x)=0.5-0.25x$. $\hat{f}(-0.03)=0.5075$. In this case, $f(-0.03)\approx 0.5068$.

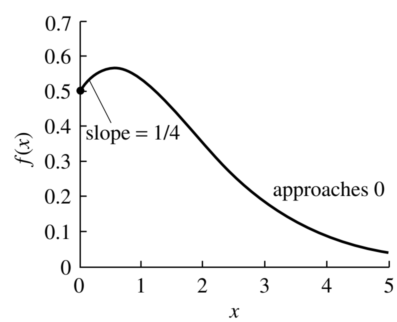
7. $f'(x)=(2-x^2)e^{-x}$, which is 0 at $x=\sqrt{2}$. Because $f(0)=0$ and $\lim_{x\to\infty}f(x)=0$ (the exponential function declines faster than the quadratic function increases), this must be a maximum.



9. $\hat{f}(x)=1+x-x^2$.

11. $\hat{g}(x)=\frac{1}{2}+\frac{1}{4}x-\frac{1}{4}x^2$.

13.

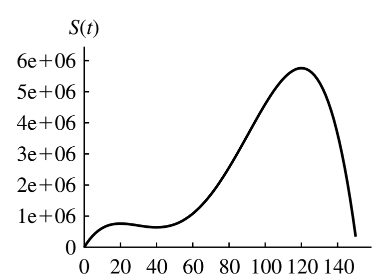


15. a. $S'(t)=-t^3+180t^2-8000t+96000$, which is 0 at $t=20$, $t=40$, and $t=120$.

b. Substituting the endpoints ($t=0$ and $t=150$) and the critical points into the function $S(t)$, we find a maximum of 576,000 at $t=120$.

c. $S''(t)=-3t^3+360t-8000$. Then $S''(20)=-2000$, $S''(40)=1600$, and $S''(120)=-8000$. The first and last are maxima, and the middle one is a minimum.

d.



17. a. $c_{t+1}=0.75(1-\alpha)c_t+0.25\gamma$.

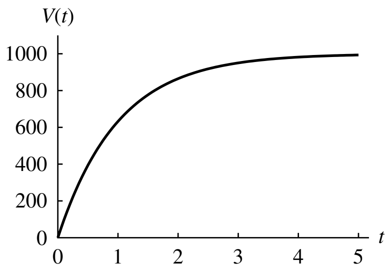
b. $c^*=0.25\gamma/(0.25+0.75\alpha)$.

c. $\alpha c^*=0.25\alpha\gamma/(0.25+0.75\alpha)$.

d. This has its maximum at $\alpha=1$.

e. Sure, absorb as much as you can as fast as you can.

19. a.



b. The fraction outside is $1 - H(t) = 1/(1 + e^t)$, so the total volume outside is

$$\frac{1000(1 - e^{-t})}{(1 + e^t)} \ \mu\text{m}^3$$

Call this function $V_o(t)$.

c.

$$V_o'(t) = \frac{1000(2 + e^{-t} - e^t)}{(1 + e^t)^2} \ \frac{\mu\text{m}^3}{\text{days}}.$$

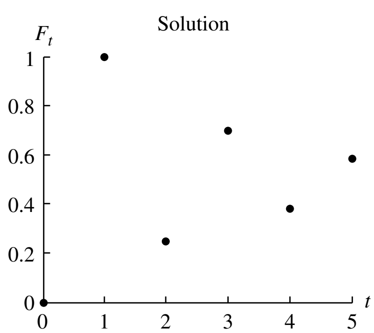
d. This is a bit tricky. The maximum is the critical point where $V_o'(t) = 0$, or where $2 + e^{-t} - e^t = 0$. Letting $x = e^t$, this is $2 + 1/x - x = 0$. Multiplying both sides by x, we get the quadratic $2x + 1 - x^2 = 0$, which can be solved with the quadratic formula to give $x = 1 + \sqrt{2}$, so that $t = \ln(1 + \sqrt{2}) \approx 0.88$ days.

21. a. The more food there is, the more is eaten, up to a limit of b. The replenishment is enough to refill the table.

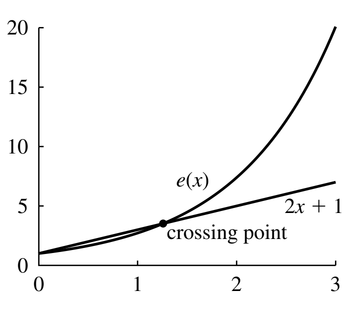
b. Full after 5 min, 25% full after 10 min.

c. The equilibrium is at $F = 0.5$.

d. $G(0.6) = 0.4375$.



23. a. It looks like $x_0 = 1$ is a good guess.



b. We need to solve $g(x) = e^x - 2x - 1 = 0$, so

$$x_{t+1} = x_t - \frac{g(x_t)}{g'(x_t)} = x_t - \frac{e^x - 2}{e^x - 2x - 1}$$

c. $x_1 \approx 1.3922$.

d. The derivative of the updating function is

$$\frac{e^x(e^x - 2x - 1)}{(e^x - 2)^2}$$

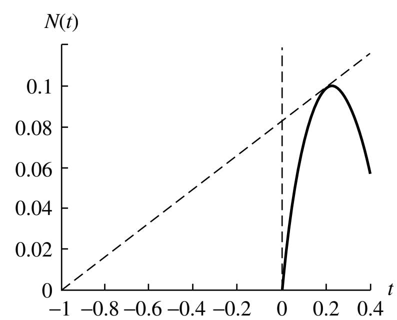
which is 0 when $e^x - 2x - 1 = 0$.

25. a. Let $N(t)$ be net energy gain. Then $N(t) = F(t) - 2t$.

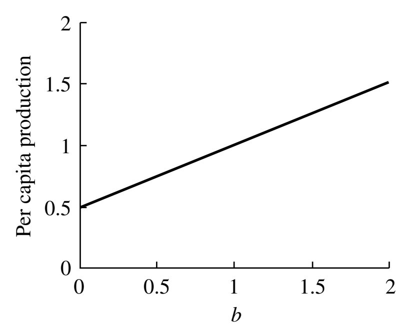
b. At $t = \sqrt{3/2} - 1 \approx 0.225$.

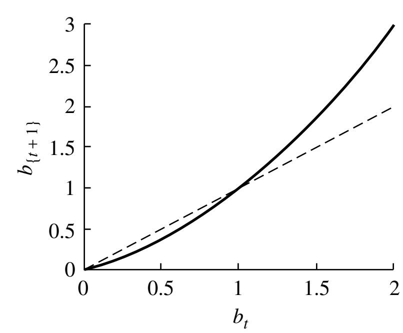
c. We need to maximize $\frac{N(t)}{1+t}$, which occurs when $t = 0.2$.

d. The answer to part **c** is smaller because the bee has other options besides sucking as much nectar as possible out of the flower.



27. a.





b. The updating function is $b_{t+1} = rb_t\left(1 + \frac{b_t}{K}\right)$.

c. Equilibria at $b^* = 0$ and $b^* = 10^6$.

d. The Equilibrium at $b^* = 0$ is stable, and the equilibrium at $b^* = 10^6$ is unstable.

29. a. The updating function f is $f(x) = 4x^2/(1 + 3x^2)$, multiplying the per capita production by x, the number of individuals.

b. First, find that 0 is a solution. The rest is a quadratic, which has solutions at $x = 1/3$ and $x = 1$.

c. We find that $f'(x) = 8x/(1 + 3x^2)^2$. Then $f'(0) = 0$, $f'(1/3) = 3/2$, and $f'(1) = 1/2$. Therefore, the equilibria at 0 and 1 are stable and the one at $1/3$ is unstable.

d. The tangent line at 0 is $\hat{f}(x) = 0$. The tangent line at $x = 1/3$ is $\hat{f}(x) = 1/3 + 3/2(x - 1/3)$. The tangent line at $x = 1$ is $\hat{f}(x) = 1 + 1/2(x - 1)$.

e. This dynamical system shoots off to infinity for starting points greater than 1/3 and shoots off to negative infinity for starting points less than 1/3. This is different from the behavior of the original system, in which such solutions approach $x = 1$ and $x = 0$, respectively.