Math 155 **Exam 2** **Spring 2007**

NAME: Solutions

SECTION: ____________ TIME: ____________

INSTRUCTOR: ____________________

Instructions: The exam is closed book and closed notes. You may use an approved calculator, but be sure to show your work on each problem for full credit. Work that is crossed out or erased will not be graded. Turn in any scratch paper that you use during the exam. You will have one hour and 45 minutes to work on the exam.

Problem	Points	Score
1	20	
2	16	
3	~~12~~ 14	
4	~~14~~ 12	
5	8	
6	16	
7	14	
Total	100	

1. (20 pts) Find the derivatives of the following functions. You do not have to simplify your answers. Be sure to use parentheses to indicate multiplication where appropriate.

(a) $f(x) = (1 + x^2)^4$

$f'(x) = 4(1+x^2)^3(2x)$

(b) $f(t) = 2^t \cdot t^2$

$f'(t) = 2^t \ln 2 \cdot t^2 + 2^t \cdot 2t$

(c) $f(x) = \dfrac{\ln(x)}{3x + \sin(e^x)}$

$$f'(x) = \frac{\frac{1}{x}(3x + \sin(e^x)) - \ln(x)(3 + \cos(e^x)e^x)}{[3x + \sin(e^x)]^2}$$

(d) $f(x) = (cx^b)(e^{ax^2})$

$f'(x) = (cbx^{b-1}) \cdot (e^{ax^2}) + (cx^b)(e^{ax^2})(2ax)$

(e) $f(t) = \sin(\cos(4\ln(t)))$

$f'(t) = \cos(\cos(4\ln t)) \cdot -\sin(4\ln(t)) \cdot \frac{4}{t}$

2. (16 pts) Evaluate the following limits. Show all of your work. If you use L'Hospital's Rule, justify why it can be applied each time you use it. If you use knowledge of leading behaviors, justify your work by explaining all of your steps.

(a) $\displaystyle\lim_{x\to\infty} \frac{e^{5x}+2}{x^2+1}$

Leading behavior

$\displaystyle\lim_{x\to\infty} \frac{e^{5x}+2}{x^2+1} = \lim_{x\to\infty} \frac{e^{5x}}{x^2} = \infty$

because exponentials grow to ∞ faster than powers

Form $\frac{\infty}{\infty}$ so use L'Hospital's Rule:

$\displaystyle = \lim_{x\to\infty} \frac{5e^{5x}}{2x} = \lim_{x\to\infty} \frac{25e^{5x}}{2} = \infty$

again form $\frac{\infty}{\infty}$ by L'Hospital's

(b) $\displaystyle\lim_{x\to\infty} \frac{\ln(2x)+7x}{x^2+\sqrt{x}+4}$

leading behavior

let $f(x) = \ln(2x) + 7x \quad f_\infty(x) = 7x$

let $g(x) = x^2 + \sqrt{x} + 4 \quad g_\infty(x) = x^2$

So consider $\displaystyle\lim_{x\to\infty} \frac{7x}{x^2} = \lim_{x\to\infty} \frac{7}{x} = 0$

(c) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

L'Hospital's:
Form $\frac{0}{0}$

$\displaystyle = \lim_{x\to 1} \frac{\frac{1}{2}x^{-1/2}}{1} = \frac{1}{2}(1)^{-1/2} = \boxed{\frac{1}{2}}$

Factor:

$\displaystyle \lim_{x\to 1} \frac{\cancel{\sqrt{x}-1}}{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)}$

$\displaystyle = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$

(d) $\displaystyle\lim_{x\to\infty} \frac{\ln(e^{3x})+4x^{1/2}}{x^2+1} = \lim_{x\to\infty} \frac{3x+4x^{1/2}}{x^2+1}$

Leading behavior

$\displaystyle = \lim_{x\to\infty} \frac{3x}{x^2} = \lim_{x\to\infty} \frac{3}{x} = 0$

L'Hospitals
Form $\frac{\infty}{\infty}$

$\displaystyle = \lim_{x\to\infty} \frac{3+2x^{-1/2}}{2x}$

$\displaystyle = \lim_{x\to\infty} \frac{3}{2x} + \frac{1}{x\sqrt{x}} = 0$

or unsimplified L'Hospital's
Form $\frac{\infty}{\infty}$

$\displaystyle \lim_{x\to\infty} \frac{\frac{1}{e^{3x}}\cdot e^{3x}\cdot 3 + 2x^{-1/2}}{2x}$

$\displaystyle = \lim_{x\to\infty} \frac{3+2x^{-1/2}}{2x}$ etc.

3. (14 points) Consider the discrete-time dynamical system:

$$x_{t+1} = \frac{x_t(x_t - r)}{3 + r} + r, \qquad r > 0$$

(a) Verify that $x^* = r$ and $x^* = 3 + r$ are equilibria.

$r = \frac{r(r-r)}{3+r} + r$

$r = 0 + r$

$r = r$ ✓

$3+r = \frac{\cancel{(3+r)}(3+r-r)}{\cancel{3+r}} + r$

$3+r = 3 + 0 + r$

$3+r = 3+r$ ✓

(b) Show that the derivative of the updating function is $\frac{1}{3+r}(2x - r)$.

$f(x) = \frac{x(x-r)}{3+r} + r$

$= \frac{x^2 - rx}{3+r} + r$

$f'(x) = \frac{1}{3+r} \cdot (2x - r)$

or by quotient:

$f'(x) = \frac{(2x-r)(3+r) - (x^2 - rx)(0)}{(3+r)^2} + 0$

$= \frac{2x-r}{3+r}$

(c) Let $r = 2$. Use the Slope Criterion/Stability Test to determine the stability of each equilibrium.

If $r = 2$, equilibria are $x^* = 2, 2+3$ or $x^* = 2, 5$

Test $x^* = 2$: $|f'(2)| = \left|\frac{1}{3+2}(2(2) - 2)\right| = \left|\frac{1}{5}(2(2)-2)\right| = \left|\frac{1}{5} \cdot 2\right| = \left|\frac{2}{5}\right| = \frac{2}{5} < 1$

$x^* = 2$ is stable

Test $x^* = 5$: $|f'(5)| = \left|\frac{1}{5}(2(5)-2)\right| = \left|\frac{1}{5}(8)\right| = \left|\frac{8}{5}\right| = \frac{8}{5} > 1$

$x^* = 5$ is unstable

4. (12 pts) Let $f(x) = \dfrac{ax^3 + x^{-1}}{x^2 + c}$, and $a,\ c > 0$.

(a) Find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$.

as $x \to \infty$

$\underbrace{ax^3}_{\to\infty} + \underbrace{x^{-1}}_{\to 0}$ $\quad \underbrace{x^2}_{\to\infty} + \underbrace{c}_{\to c}$

$f_\infty(x) = \dfrac{ax^3}{x^2} = ax$

(b) Find $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$.

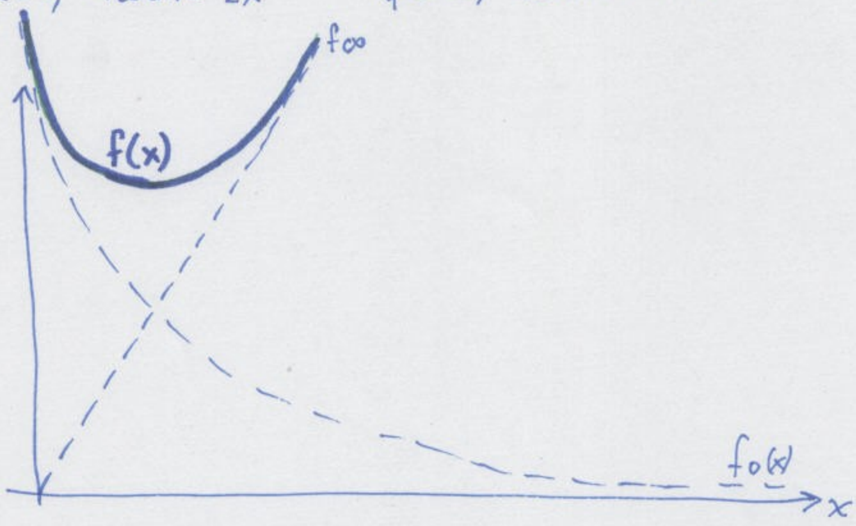
as $x \to 0$

$\underbrace{ax^3}_{\to 0} + \underbrace{x^{-1}}_{\to\infty}$ $\quad \underbrace{x^2}_{\to 0} + \underbrace{c}_{\to c}$

$f_0(x) = \dfrac{x^{-1}}{c} = \dfrac{1}{c} \cdot \dfrac{1}{x}$

(c) Let $a = 2$ and $c = 4$. Use the method of matched leading behaviors to sketch a graph of $f(x)$ on the interval $x \geq 0$. Label your axes and indicate where you have graphed $f_0(x)$, $f_\infty(x)$, and $f(x)$.

if $a=2$, $f_\infty(x) = 2x$ $\qquad$ if $c=4$, $f_0(x) = \frac{1}{4} \cdot \frac{1}{x}$



5. (8 points) For each item below, fill in the blank with the appropriate letter. Each blank gets one letter only.

Statement	Response
$x = c$ is an equilibrium if i	(a) $f'(c) = 0$
$x = c$ is a stable equilibrium if l	(b) $f'(c) > 0$
$x = c$ is a critical point if a or if d	(c) $f'(c) < 0$
$f(c)$ is a local maximum if a and g	(d) $f'(c)$ undefined
$f(x)$ is increasing at $x = c$ if b	(e) $f''(c) = 0$
$f(x)$ is concave down at $x = c$ if g	(f) $f''(c) > 0$
	(g) $f''(c) < 0$
	(h) $f(c) = 0$
	(i) $f(c) = c$
	(j) $\lvert f'(c)\rvert < 0$
	(k) $\lvert f'(c)\rvert > 0$
	(l) $\lvert f'(c)\rvert < 1$
	(m) $\lvert f'(c)\rvert > 1$
	(n) $\lvert f'(c)\rvert = 1$

6. (16 points) Let $f(x) = \frac{2}{(x+2)}$.

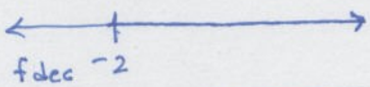
(a) Find the critical points.

$$f'(x) = \frac{0(x+2) - 2(1)}{(x+2)^2} = \frac{-2}{(x+2)^2}$$

C.P.: $f'(x) \neq 0$

$f'(x)$ DNE if $(x+2)^2 = 0$ or if $x = -2$

(b) Write out the intervals of increase and decrease. Justify your answers with calculus.



$$f'(-4) = \frac{-2}{(-4+2)^2} < 0$$

$$f'(0) = \frac{-2}{(2)^2} < 0$$

$\boxed{f \text{ is decreasing on } (-\infty, -2) \text{ and } (-2, \infty)}$

(c) Write the intervals of concave up and concave down. Justify your answers with calculus.

$$f''(x) = \frac{0 - (-2)(2(x+2)(1))}{(x+2)^4} = \frac{4x+8}{(x+2)^4}$$

$f''(x) = 0$ if $4x + 8 = 0$

$x = -2$

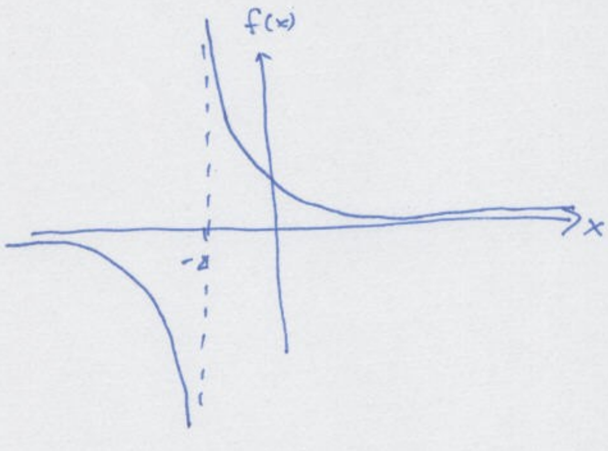
$$f''(-4) = \frac{4(-4)+8}{(-4+2)^4} < 0 \text{ CD}$$

$$f''(0) = \frac{8}{(0+2)^4} > 0 \text{ CU}$$

f is concave up on $(-2, \infty)$

f is concave down on $(-\infty, -2)$

(d) Use the information above to sketch a graph of $f(x)$.



7. (14 pts) Consider the function $f(x)$ on the interval $[0, 10]$ where

$$f(x) = x^3 - 9x^2 + 15x - 5$$

a) Find the critical points of $f(x)$.

$f'(x) = 3x^2 - 18x + 15$

$0 = 3x^2 - 18x + 15$

$0 = x^2 - 6x + 5$

$0 = (x-1)(x-5)$

$\boxed{x = 1, 5}$

b) Classify each critical point as a local minimum, local maximum or neither. Justify your answer by using either the first or second derivative test.

1 5

$f'(0) = 15$ f inc

$f'(2) = 3(2)^2 - 18(2) + 15 < 0$ f dec

$f'(10) = 3(10)^2 - 18(10) + 15 > 0$ f inc

local max local min

$x = 1$ gives a local maximum

$x = 5$ gives a local minimum

c) Find the global maximum and global minimum on $[0, 10]$ and give both the x and y coordinates of each.

$f(0) = -5$

$f(1) = 2$

$f(5) = -30$ – min

$f(10) = 245$ – max

Global Maximum at $(10, 245)$

Global Minimum at $(5, -30)$