M155 Exam 2 Solutions Spring 2006

1. (16 pts) Find the derivatives of the following functions. You do not have to simplify your answers. Be sure to use parentheses to indicate multiplication where appropriate.

a) $f(x) = x^9 e^{-3x^2-32}$.

$$f'(x) = \left(9x^8 e^{-3x^2-32}\right)\left(x^9 e^{-3x^2-32}(-6x)\right)$$

b) $f(x) = \frac{11}{x^{-4}-6x+13}$.

$$f'(x) = \frac{-11(-4x^{-5}-6)}{(x^{-4}-6x+13)^2}$$

c) $f(x) = 5\ln(\cos x)$.

$$f'(x) = 5 \cdot \frac{1}{\cos x} \cdot (-\sin x)$$

d) $f(x) = (\cos 2\pi x)(\sin \pi x)$.

$$-\sin(2\pi x) \cdot 2\pi \cdot \sin \pi x + \cos(2\pi x) \cdot \cos(\pi x) \cdot \pi$$

2. (20 pts) Given the function $f(x) = \frac{x^2}{2x^2+6x+2}$ on the interval $(-\infty, \infty)$

a) Find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$, and $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$

f_∞ : Top: x^2

bottom: $2x^2$

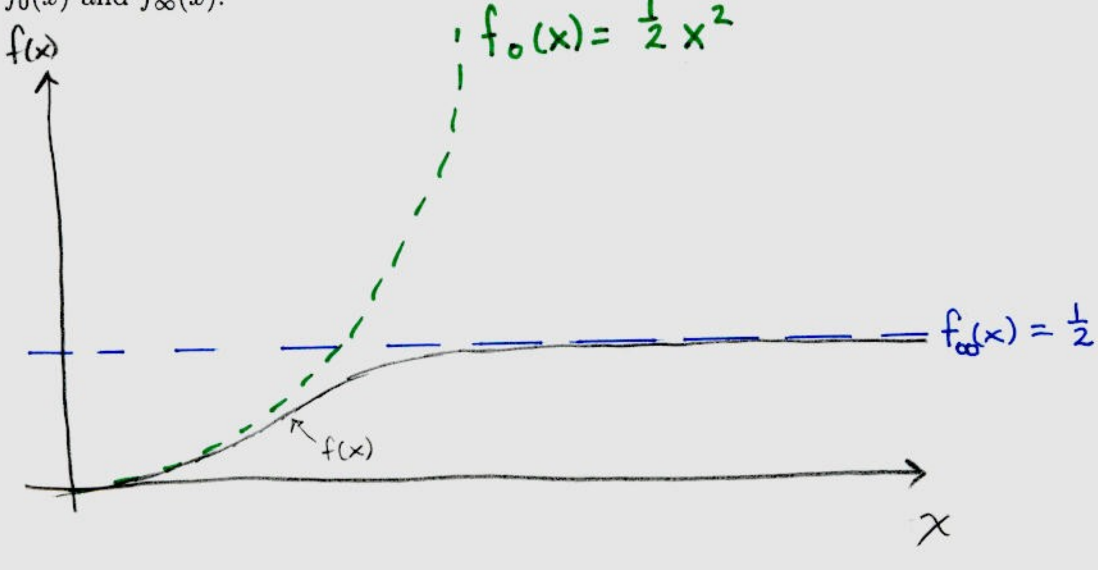
$$f_\infty = \frac{x^2}{2x^2} = \frac{1}{2}$$

f_0 : Top: x^2

bottom: 2

$$f_0 = \frac{x^2}{2} = \frac{1}{2}x^2$$

b) Use the method of matched leading behaviors to sketch a graph of $f(x)$ on the interval $[0, \infty)$. Label your axes and indicate where you have graphed $f_0(x)$ and $f_\infty(x)$.



c) Find the intervals where $f(x)$ is increasing and decreasing on $(-\infty, \infty)$.

$$f'(x) = \frac{2x(2x^2+6x+2) - x^2(4x+6)}{(2x^2+6x+2)^2} = \frac{4x^3+12x^2+4x-4x^3-6x^2}{(2x^2+6x+2)^2}$$

$$f'(x) = \frac{6x^2+4x}{(2x^2+6x+2)^2}$$

$f'(x)$ is undefined if $2x^2+6x+2=0$

$$x^2+3x+1=0$$

$$x = \frac{-3 \pm \sqrt{3^2-4}}{2} = \frac{-3\pm\sqrt{5}}{2}$$

$x = -2.618,\ -.381966$

$f'(x)=0$ if $6x^2+4x=0$

$x(6x+4)=0$

$\boxed{x=0}$ or $6x+4=0$

$6x=-4$

$\boxed{x=-2/3}$

$=-.66\bar{6}$

$f'(-3)>0$ $\quad f'(-1)>0$ $\quad f'(-.5)<0$ $\quad f'(2)>0$

-2.618 $\quad -.6\bar{6}$ $\quad -.38$ $\quad 0$

$f'(-.2)<0$

$f(x)$ increases on
$(-\infty,-2.618) \cup (-2.618,-.\bar{6}) \cup (0,\infty)$
$f(x)$ decreases on
$(-.\bar{6},-.38) \cup (-.38,0)$

d) Find any local minima or local maxima of $f(x)$ on $(-\infty, \infty)$.

$f(-.\bar{6}) = -.4$ local maximum

$f(0)=0$ Local minimum

e) Do you think your graph from part b is reasonably accurate based on your answers to parts c and d? Why or why not?

Yes, as $f(x)$ should have a low point at $(0,0)$, and then increase with out any other critical points

3. (12 pts) A runner's arm swings rhythmically according to the equation

$$y(t) = \frac{\pi}{8}\cos[5\pi(t - 2/5)]$$

where t is time in sec, and $y(t)$ is the angle between the actual position of the upper arm and the downward vertical position.

a. Calculate the velocity $y'(t)$ and acceleration $y''(t)$.

$$y'(t) = \frac{\pi}{8}\left(-\sin(5\pi(t - 2/5))\right)(5\pi) = -\frac{5\pi^2}{8}\sin(5\pi(t - 2/5))$$

$$y''(t) = -\frac{5\pi^2}{8}\cos(5\pi(t - 2/5))(5\pi) = -\frac{25\pi^3}{8}\cos(5\pi(t - 2/5))$$

b. Give two times when the velocity is zero. What is happening at these times?

$$y'(t) = 0 \text{ if } -\frac{5\pi}{8}\sin(5\pi(t - 2/5)) = 0$$

$$\sin(5\pi(t - 2/5)) = 0$$

$$\sin^{-1}(\sin(5\pi(t - 2/5))) = \sin^{-1}(0)$$

$$5\pi(t - 2/5) = 0$$

$$\boxed{t = 2/5}$$

→ another solution if $\sin^{-1}(0) = \pi$,

so $5\pi(t - 2/5) = \pi$

$\boxed{t = 1/5}$

If the velocity is 0, The arm isn't moving - probably the arm is at an end of motion, changing directions.

c. Is the function $y(t)$ concave up or concave down when $t = 5$ s? (Justify your answer) What does this tell you about the runner's arm at this time?

$$y''(5) = \frac{-25\pi^3}{8}\cos(5\pi(5 - 2/5)) = -\frac{25\pi^3}{8}\cos(23\pi) = \frac{25\pi^3}{8} > 0$$

$\therefore$ $y(t)$ is concave up.

This tells us that the arm is accelerating and picking up speed.

4. (16 pts) Evaluate the following limits. Show all of your work. If you use L'Hospital's Rule, justify why it can be applied each time you use it. If you use knowledge of leading behaviors, justify your work.

a) $\lim_{x\to 2} \frac{x^2+3x-10}{x^2-4}$ $= \frac{2^2+3(2)-10}{2^2-4} = \frac{0}{0}$ Form $\frac{0}{0}$

use L'Hopital's Rule: $\lim_{x\to 2} \frac{2x+3}{2x} = \frac{2(2)+3}{2(2)} = \boxed{\frac{7}{4}}$

b) $\lim_{x\to 0} \frac{\tan x}{\sin x}$ Form $\frac{0}{0}$

use L'Hopital's Rule: $\lim_{x\to 0} \frac{\sec^2 x}{\cos x} = \lim_{x\to 0} \frac{\frac{1}{\cos^2 x}}{\cos x}$

$= \lim_{x\to 0} \frac{1}{\cos^3 x} = \frac{1}{1} = \boxed{1}$

c) $\lim_{x\to\infty} \frac{e^{-3x}+4x}{x^2+e^x}$

let $f(x) = e^{-3x}+4x$ $\quad f_\infty(x) = 4x$ - use $\lim_{x\to\infty} \frac{4x}{e^x} = 0$

let $g(x) = x^2+e^x$ $\quad g_\infty(x) = e^x$

because e^x grows faster than any power function.

d) $\lim_{x\to\infty} \frac{\ln(2x+13)}{\sqrt{x+20}}$ Form $\frac{\infty}{\infty}$

By L'Hopital's Rule: $\lim_{x\to\infty} \frac{\frac{1}{2x+13}(2)}{\frac{1}{2}(x+20)^{-1/2}(1)} = \lim_{x\to\infty} 4 \cdot \frac{(x+20)^{1/2}}{2x+13}$ Form $\frac{\infty}{\infty}$,

by L'Hopital's Rule: $4 \lim_{x\to\infty} \frac{\frac{1}{2}(x+20)^{-1/2}}{2} = \lim_{x\to\infty} \frac{1}{\sqrt{x+20}} = \boxed{0}$

5. (12 pts) Suppose the amount of a certain chemical absorbed by the lung is

$$A(c) = 0.002ce^{-\alpha c}, \quad \alpha > 0.$$

a) Find the concentration $c \geq 0$ of chemical for which the amount absorbed is maximal. Justify that you have found the maximum.

$A'(c) = 0.002e^{-\alpha c} + 0.002c\, e^{-\alpha c} \cdot (-\alpha)$

Critical Points: $\underbrace{0.002 e^{-\alpha c}}_{\text{never zero}} (1 - \alpha c) = 0$

$1 - \alpha c = 0$

$1 = \alpha c$

$\boxed{c = \frac{1}{\alpha}}$

$A''(c) = 0.002 e^{-\alpha c}(-\alpha) + 0.002 e^{-\alpha c}(-\alpha) + 0.002c e^{-\alpha c}(-\alpha)(-\alpha)$

$= 0.002 e^{-\alpha c}(-\alpha)\left[2 - \alpha c\right]$

$A''(\frac{1}{\alpha}) = \underbrace{0.002 e^{-\alpha(\frac{1}{\alpha})}}_{>0} \underbrace{(-\alpha)}_{<0} \underbrace{(2-1)}_{>0} < 0$ ∴ Concave down and $c = \frac{1}{\alpha}$ gives a max

b) What is the amount absorbed for very high concentrations? (Find $\lim_{c\to\infty} A(c)$. Hint: $A(c) = \frac{0.002c}{e^{\alpha c}}$). Do you think this amount is realistic? Why or why not?

$\lim_{c \to \infty} 0.002c\, e^{-\alpha c} = \lim_{c \to \infty} \frac{0.002c}{e^{\alpha c}} = 0$

because exponential functions grow faster than power functions.

Yes, because the blood becomes saturated by the chemical and the lung cannot absorb any more chemical.

6. (12 pts) The following updating function describes a crowded population N_n.

$$N_{n+1} = \frac{\lambda N_n}{1 + aN_n}, \quad a, \lambda > 0$$

a) Find all equilibria of the updating function.

$$N^* = \frac{\lambda N^*}{1 + aN^*}$$

$$N^*(1 + aN^*) = \lambda N^*$$

$$N^* + aN^{*2} - \lambda N^* = 0$$

$$N^*(1 + aN^* - \lambda) = 0$$

$$\boxed{N^* = 0} \quad \text{or}$$

$$1 - \lambda + aN^* = 0$$

$$1 - \lambda = -aN^*$$

$$-\frac{1-\lambda}{a} = N^*$$

$$\boxed{\frac{\lambda - 1}{a} = N^*}$$

b) Use the stability test to determine under what conditions each equilibrium is stable.

$$\text{let } f(N) = \frac{\lambda N}{1 + aN}$$

$$f'(N) = \frac{\lambda(1 + aN) - \lambda N(a)}{(1 + aN)^2} = \frac{\lambda}{(1 + aN)^2}$$

$$|f'(N^*)| = |f'(0)| = \left|\frac{\lambda}{(1+0)^2}\right| = |\lambda|$$

$N^* = 0$ is stable if $\boxed{0 < \lambda < 1}$ ($\lambda > 0$)

$$\left|f'\left(\frac{\lambda-1}{a}\right)\right| = \left|\frac{\lambda}{\left(1 + a\left(\frac{\lambda-1}{a}\right)\right)^2}\right| = \left|\frac{\lambda}{(1 + \lambda - 1)^2}\right| = \left|\frac{\lambda}{\lambda^2}\right| = \left|\frac{1}{\lambda}\right|$$

since $\lambda > 0$, $\left|\frac{1}{\lambda}\right| = \frac{1}{\lambda}$

$N^* = \frac{\lambda - 1}{a}$ is stable if $\frac{1}{\lambda} < 1$ or if $\boxed{1 < \lambda}$

7. (12 pts) Consider the function $f(x)$ on the interval $[-3, 3]$ where

$$f(x) = x^3 - 12x$$

a) Find the critical points of $f(x)$ on $[-3, 3]$.

$f'(x) = 3x^2 - 12$

$3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4$

$\boxed{x = \pm 2}$

b) Classify each critical point on $[-3, 3]$ as a local minimum, local maximum or neither. Justify your answer by using either the first or second derivative test.

1^{st} derivative test:

$f'(0) < 0$

$-2 \quad 0 \quad 2$

$f'(-3) > 0$ $\quad$ $f'(3) > 0$

local max at $x = -2$

Local min at $x = 2$

(or)

2nd derivative test:

$f''(x) = 6x$

$f''(-2) < 0$ concave down means Local max at $x = -2$

$f''(2) > 0$ concave up means Local min at $x = -2$

c) Find the global maximum and global minimum on $[-3, 3]$ and give both the x and y coordinates of each.

$f(-3) = 9$

$\boxed{f(-2) = 16}$ Global max $(-2, 16)$

$\boxed{f(2) = -16}$ global min $(2, -16)$

$f(3) = -9$