M155 Exam 1 Spring 2006 Solutions

1. (16 pts) The women's Olympic record for the 1500 m race in 1972 was 4 min. 1.4 s and in 1988 was 3 min. 53.9 s.

a) Convert these times to seconds and find the equation of the line in slope-intercept form connecting these points.

4 minutes, 1.4 seconds = $4(60) + 1.4 = 241.4$ seconds

3 minutes, 53.9 seconds = $3(60) + 53.9 = 233.9$ seconds

Let $t=0$ be 1972

Points: (year, seconds)

$(0, 241.4)$ and $(16, 233.9)$

Slope: $\dfrac{241.4 - 233.9}{0-16} = -.46875$

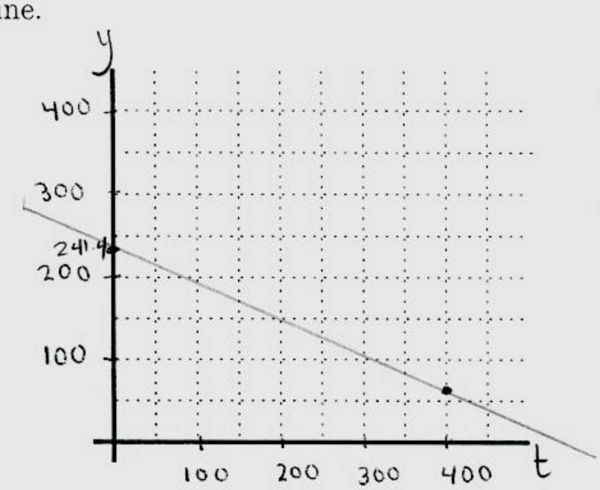
Point-slope form:

$y - 241.4 = -.46875(t-0)$

slope intercept form:

$$\boxed{y = -.46875t + 241.4}$$

b) Graph the line.



y intercept = 241.4

another point:

$y(400) = -.46875(400) + 241.4$

$= 53.9$

c) If things continue at this rate, when would the women finish in exactly no time?

(Find t-intercept)

$0 = -.46875t + 241.4$

$-241.4 = -.46875t$

$t = \dfrac{241.4}{.46875} = 514.98\bar{6}$

in about 515 years, or the year $1972 + 515 = 2487$

2. (12 pts) A population of bacteria doubles every hour, but 1.0×10^6 individuals are removed after reproduction to be converted into valuable biological by-products.

a) Write a discrete-time dynamical system to describe the population.

$$b_{t+1} = 2\,b_t - 1.0 \times 10^6$$

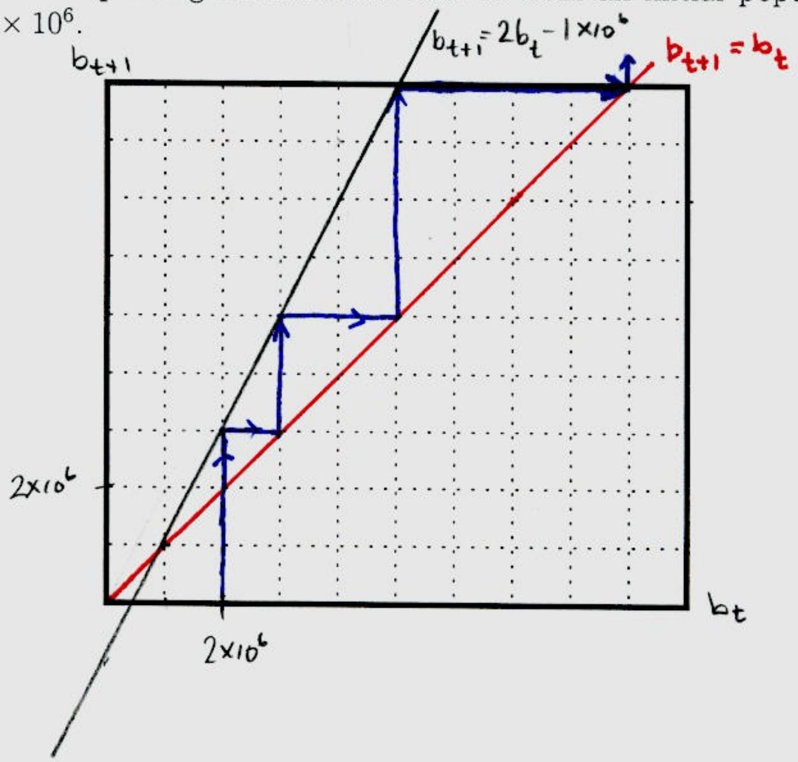
b) Find the equilibrium solution.

$$b^* = 2b^* - 1.0 \times 10^6$$

$$-b^* = -1.0 \times 10^6$$

$$b^* = 1.0 \times 10^6$$

c) Graph the updating function and cobweb from an initial population of $b_0 = 2.0 \times 10^6$.



3. (10 pts) Suppose the size of a population of bacteria at time t is given by $b(t) = 1.5t + 6$ where t is measured in hours.

a) Find the average rate of change of the population between times t and $t + \Delta t$. Justify your answer.

Av. ROC: $\dfrac{b(t+\Delta t) - b(t)}{\Delta t}$

$$= \frac{[1.5(t+\Delta t)+6] - [1.5t+6]}{\Delta t} = \frac{1.5\Delta t}{\Delta t} = 1.5 \text{ bacteria/hour}$$

OR $b(t)$ is linear, so average rate of change = slope which is 1.5.

b) Suppose $t = 2$ and $\Delta t = 0.1$. Find the equation of the secant line in slope-intercept form between times t and $t + \Delta t$.

Slope $= 1.5$

1st point: $(2, b(2))$ or $(2, 9)$

2nd point: $(2+0.1, b(2+0.1))$ or $(2.1, 9.15)$

point slope form: $y - 9 = 1.5(t-2)$

slope intercept form: $y = 1.5t + 6$

4. (12 pts) The amount of C^{14} left t years after the death of an organism is given by

$$Q(t) = Q_0 e^{-0.000122t}$$

where Q_0 is the amount left at the time of death.

a) When will one tenth of the original amount of C^{14} be left in the remains?

$Q(t)$ = amount at time t

Q_0 = amount at time of death = original amount

When will $Q(t) = \frac{1}{10} Q_0$?

$$0.1 Q_0 = Q_0 e^{-0.000122t}$$

$$0.1 = e^{-0.000122t}$$

$$\ln 0.1 = -0.000122t$$

$$t = \frac{\ln 0.1}{-0.000122}$$

$$t = 18{,}873.6483$$

b) If 2.5×10^{-8} grams are present after 10,000 hours, how much C^{14} was originally present in the organism?

$2.5 \times 10^{-8} = Q(10{,}000)$ Find Q_0

$$2.5 \times 10^{-8} = Q_0 e^{-0.000122(10{,}000)}$$

$$2.5 \times 10^{-8} = Q_0 e^{-1.22}$$

$$2.5 \times 10^{-8} = Q_0 (.29523)$$

$$Q_0 = \frac{2.5 \times 10^{-8}}{.29523} = \boxed{8.46797 \times 10^{-8}}$$ grams of C^{14}

5. (14 pts) An organism is breathing a chemical that modifies the depth of its breaths. Suppose the discrete-time dynamical system for the concentration c_t of the chemical in the lung is

$$c_{t+1} = c_t - \frac{c_t^2}{c_t + 1} + \frac{c_t}{c_t + 1}$$

a) Find the equilibria. Show all of your work to receive credit. No points will be given for answers only.

$$\cancel{c^*} = \cancel{c^*} - \frac{c^{*2}}{c^*+1} + \frac{c^*}{c^*+1}$$

$$0 = \frac{-c^{*2} + c^*}{c^*+1}$$

So $0 = -c^{*2} + c^*$

$0 = c^*(-c^* + 1)$

$\boxed{c^* = 0}$

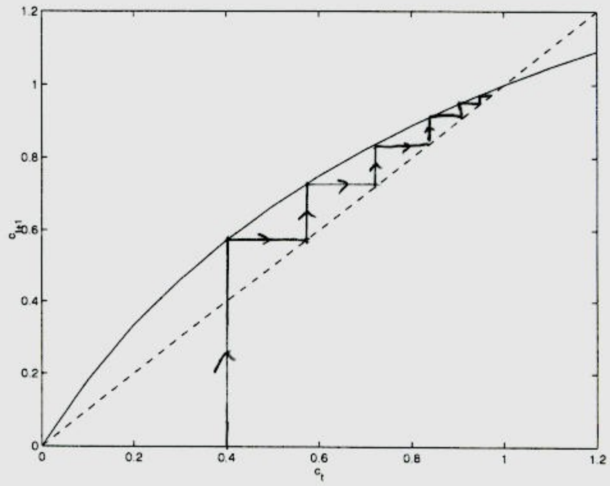
OR

$-c^* + 1 = 0$

$-c^* = -1$

$\boxed{c^* = 1}$

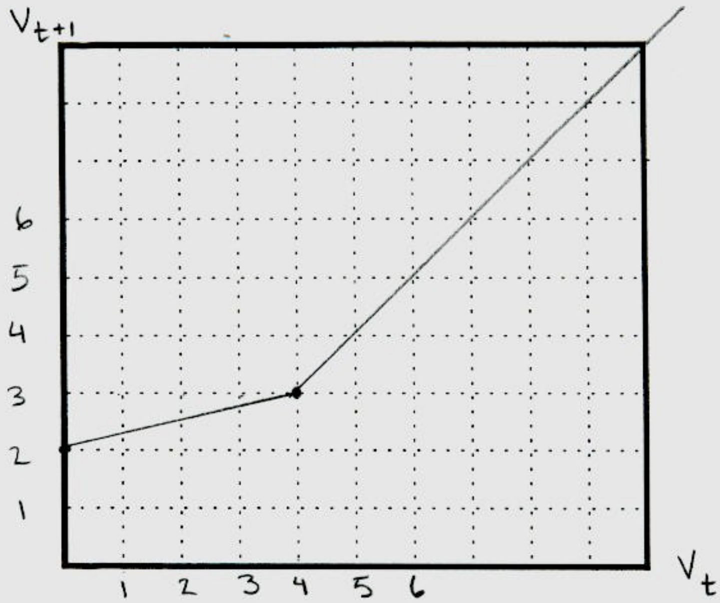
b) Draw a cobweb diagram starting from the initial concentration $c_0 = .4$. What is the long-term concentration of the chemical in the lung?



Long term concentration is the equilibrium: $c^* = 1$

6. (12pts) Accurately graph the updating function

$$V_{t+1} = \begin{cases} \frac{1}{4}V_t + 2, & \text{if } V_t \le 4 \\ V_t - 1, & \text{if } V_t > 4 \end{cases}$$



b) Is the function continuous? Why or why not?

The function is continuous because the limits from the left and from the right are equal to the function value at every point.

OR $\lim_{V_t \to a^+} V_{t+1} = \lim_{V_t \to a^-} V_{t+1} = V_{t+1}(a)$

c) Find $\lim_{V_t \to 3} V_{t+1}$, if it exists.

(3 < 4, so use the "top" definition only)

$\lim_{V_t \to 3} V_{t+1} = \lim_{V_t \to 3} \frac{1}{4} V_t + 2 = \frac{1}{4}(3) + 2 = 2.75$

or $\frac{11}{4}$

7. (14 pts) a) Find the average rate of change of the function $f(x) = 6x^2 + 3$ as a function of Δx.

$$\text{Av. Roc} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{[6(x+\Delta x)^2 + 3] - [6x^2+3]}{\Delta x}$$

$$= \frac{6(x^2 + 2x\Delta x + \Delta x^2) + 3 \quad -6x^2 - 3}{\Delta x}$$

$$= \frac{12x\Delta x + 6\Delta x^2}{\Delta x}$$

$$= \boxed{12x + 6\Delta x}$$

b) Find the limit in your answer to part b as $\Delta x \to 0$.

$$\lim_{\Delta x \to 0} 12x + 6\Delta x = 12x + 6(0) = \boxed{12x}$$

c) What is the instantaneous rate of change of this function at $x = 1$?

$$\text{Inst. ROC} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\text{So Inst RoC} = \lim_{\Delta x \to 0} 12x + 6\Delta x = 12x$$

$$\text{if } x=1, \text{ Inst. RoC} = 12(1) = 12$$

8. (10 pts) Find the following limits, if they exist. Show all of your work and justify your answers.

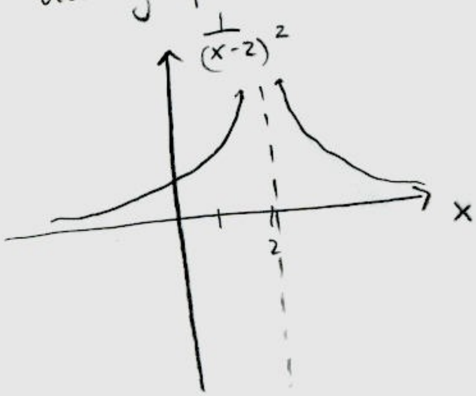
a) $\lim_{x \to 0^+} \sqrt{x} + 12$ $= \sqrt{0} + 12 = 12$

b) $\lim_{\Delta x \to 0} \frac{2(x^2 + 2x\Delta x + \Delta x^2) - 2x^2}{\Delta x}$ $= \lim_{\Delta x \to 0} \frac{2x^2 + 4x\Delta x + 2\Delta x^2 - 2x^2}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2\Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} 4x + 2\Delta x = 4x + 2(0) = \boxed{4x}$

c) $\lim_{x \to 2} \frac{1}{(x-2)^2}$

Use graph:



Thus $\boxed{\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty}$