MATH 155
Final Exam
Spring 2013

1. (14 points) Suppose that the population $m_t$ of macaws satisfies the discrete-time dynamical system

$$m_{t+1} = 3(1 - m_t)m_t - hm_t,$$

where $h > 0$ is a positive parameter, and $m_t$ is measured in hundreds of macaws.

(a) Find all equilibria. For what values of $h$ is there more than one equilibrium that makes biological sense?

$$m^* = 3(1-m^*)m^* - hm^*$$
$$= 3m^* - 3m^{*2} - hm^*$$
$$0 = (2-h)m^* - 3m^{*2}$$
$$= m^*(2-h-3m^*)$$
$$\Rightarrow \boxed{m^* = 0}, \text{ or } 2-h-3m^* = 0;$$
$$\boxed{m^* = \frac{2-h}{3}}$$

$\frac{2-h}{3} \geq 0$ for $h \leq 2$.

There is more than one equilibrium that makes biological sense for $(0<)\ h<2$

(b) For each equilibrium, use the Stability Theorem/Criterion to determine the values of $h$ for which that equilibrium is stable. Show clearly how you are using the Stability Theorem/Criterion.

$$m_{t+1} = f(m_t), \quad f(m) = 3(1-m)m - hm = (3-h)m - 3m^2.$$
$$f'(m) = 3-h-6m$$

$m^* = 0$ is stable if $|f'(0)| < 1$;

$$|3-h-6(0)| < 1;$$
$$-1 < 3-h < 1; \quad -4 < -h < -2;$$
$$\underline{2 < h < 4}$$

$m^* = \frac{2-h}{3}$ is stable if

$$\left|f'\left(\tfrac{2-h}{3}\right)\right| < 1;$$
$$\left|3-h-6\left(\tfrac{2-h}{3}\right)\right| < 1;$$
$$|3-h-4+2h| < 1;$$
$$|-1+h| < 1$$
$$-1 < -1+h < 1$$
$$\underline{0 < h < 2}.$$

2. (14 points) (a) Consider the function $f(x) = x^4 - 4x^2 + 2$. i) Find *all* critical points of $f(x)$. ii) Determine the global maximum and global minimum of $f(x)$ on the interval $[-1, 3]$. Justify your answer and show your work clearly for full credit.

$f(x) = x^4 - 4x^2 + 2$

$f'(x) = 4x^3 - 8x \overset{\text{set}}{=} 0$

$4x(x^2 - 2) = 0$

$x = 0, \quad x = \pm\sqrt{2}$

$$\boxed{\begin{array}{c}\text{critical pts of } f(x): \\ x = 0, \\ x = \pm\sqrt{2}\end{array}}$$

Maxima and Minima for $f(x)$ on $[-1, 3]$ can occur at

(i) endpoints

$f(-1) = -1$

$f(3) = 47$ ⟵ global max at $(x = 3, f(x) = 47)$

(ii) critical points in $[-1, 3]$:

$f(0) = 2$

$f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 2$

$= 4 - 8 + 2 = -2$ ⟵ global min at $(x = \sqrt{2}, f(x) = -2)$

(b) Suppose that the production $P$ of starch in a plant depends on time $t$ in the following manner:

$$P(t) = \frac{10t}{1 + t^2} \text{ grams per day.}$$

Find the *positive* critical point of the function $P(t)$, and use either the first or second derivative test to determine if there is either a local maximum or local minimum at that point. Show your work clearly for full credit.

$$P'(t) = \frac{(1+t^2)(10) - 10t(2t)}{(1+t^2)^2}$$

$$= \frac{10 + 10t^2 - 20t^2}{(1+t^2)^2} = \frac{10 - 10t^2}{(1+t^2)^2} \overset{\text{set}}{=} 0$$

$\Rightarrow 10 - 10t^2 = 0;$

$1 - t^2 = 0$

$t = 1$ : positive critical point

1st derivative test:

+ + + | − − −

0    1    2

$P'(0) = 10 > 0 \quad P'(1) = 0 \quad P'(2) = -3 < 0$

$\Rightarrow$ local max at $(t = 1, P(t) = 5)$

or

2nd derivative test:

$$P''(t) = \frac{(1+t^2)^2(-20t) - (10 - 10t^2)(2)(1+t^2)(2t)}{(1+t^2)^4}$$

$$= \frac{-20t(t^3 + t)}{(1+t^2)^3}$$

$$P''(1) = \frac{-20(1)(2)}{8} < 0$$

$\Rightarrow$ local max at $(t = 1, P(t) = 5)$.

3. (15 points) Evaluate the following definite and indefinite integrals. If necessary, use substitution. Show all of your work.

(a) $\int \frac{7}{x} - 8x^{\frac{9}{4}} + \pi \, dx$

$= 7\ln|x| - \frac{8x^{13/4}}{13/4} + \pi x + C$

$= 7\ln|x| - \frac{32}{13}x^{13/4} + \pi x + C$

(b) $\int \sec^2(t) + 5\cos(t)e^{\sin(t)+1} \, dt$

$= \tan(t) + 5\int \cos(t) e^{\sin t + 1} dt = \tan(t) + 5\int e^u du$

$u \overset{set}{=} \sin(t) + 1$

$du = \cos(t)\,dt$

$= \tan(t) + 5e^{\sin(t)+1} + C$

(c) $\int_0^{\sqrt{\pi}} 3t\cos(t^2)\,dt = \int_{0=u}^{\pi=u} 3\frac{1}{2}\cos(u)\,du = 3/2 \sin(u)\Big|_0^{\pi} = 3/2\left(\sin(\pi) - \sin(0)\right)$

$u \overset{set}{=} t^2$ $\quad u(0) = 0$ $\quad u(\sqrt{\pi}) = \pi$

$du = 2t\,dt$

$= 3/2(0 - 0) = 0.$

(d) Use integration by parts to evaluate $\int 5xe^{3x}\,dx$.

$u \overset{set}{=} 5x \qquad dv = e^{3x}dx$

$du = 5dx \qquad v = \frac{1}{3}e^{3x}$

$\int 5x e^{3x} dx = \frac{5}{3}xe^{3x} - 5\int \frac{1}{3}e^{3x}dx$

$= \frac{5}{3}xe^{3x} - \frac{5}{3}\frac{1}{3}e^{3x} + C = \frac{5}{3}xe^{3x} - \frac{5}{9}e^{3x} + C$

4. (14 points) Suppose that a bacterium is absorbing plutonium from its environment. At time $t = 0$, there is 0.1 mol of plutonium in the bacterium, and plutonium enters the bacterium at a rate of $\frac{1}{2+t^3} \frac{\text{mol}}{\text{hour}}$

(a) Let $p(t)$ represent the amount (mol) of plutonium in the bacterium at time $t$ (hours). Write a pure-time differential equation and an initial condition for the situation described above.

$$\frac{dP}{dt} = \frac{1}{2+t^3}$$

$$P(0) = 0.1$$

(b) Apply Euler's Method with $\Delta t = 0.25$ to estimate the amount of plutonium in the bacterium at time $t = 0.75$. Show your work clearly using a table. Give your answer to three decimal places.
(Recall the formula $\hat{p}_{\text{next}} = \hat{p}_{\text{current}} + \frac{dp}{dt}\Delta t$, or $\hat{p}(t + \Delta t) = \hat{p}(t) + p'(t)\Delta t$).

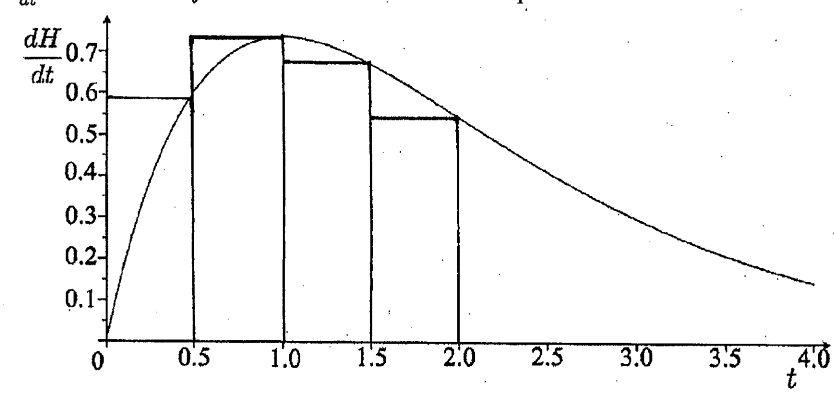
| $t$ | $\hat{P}_{\text{current}}$ | $\frac{dP}{dt} = \frac{1}{2+t^3}$ | $\hat{P}_{\text{next}} = \hat{P}_{\text{current}} + \frac{dP}{dt}\Delta t$ |
|---|---|---|---|
| 0 | 0.1 | 0.5 | 0.225 |
| 0.25 | 0.225 | 0.4961 | 0.349 |
| 0.5 | 0.349 | 0.4706 | 0.467 |
| 0.75 | $\boxed{0.467}$ | | |

5. (14 points) After a rainstorm, the growth rate of a bean stalk increases for a while, but eventually the growth rate decreases again as the ground dries up. Suppose that the rate of change of the height of the beanstalk is given by

$$\frac{dH}{dt} = te^{-t},$$

where time $t$ is measured in hours and $H(t)$ is the height (in cm) of the beanstalk $t$ hours after the rainstorm.

(a) Estimate the total change in $H(t)$ between times $t = 0$ and $t = 2$ using a right-hand Riemann Sum with $\Delta t = 0.5$. Draw your rectangles or step functions on the graph of $\frac{dH}{dt}$ below. Give your answer to three decimal places.




$$\text{R-H RS} = 0.5\, e^{-0.5}(0.5) + 1\, e^{-1}(0.5) + (1.5)e^{-1.5}(0.5) + 2\, e^{-2}(0.5)$$

$$\approx 0.638$$

(b) Find the average value of the function $f(t) = t\sin(t^2 + \pi)$ on the interval $[0, \sqrt{\pi}]$.

$$\text{Average Value} = \frac{1}{\sqrt{\pi} - 0} \int_0^{\sqrt{\pi}} t \sin(t^2+\pi)\, dt$$

$$u \overset{\text{let}}{=} t^2+\pi$$
$$du = 2t\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{u=\pi}^{u=2\pi} \frac{1}{2}\sin(u)\,du = \frac{1}{2\sqrt{\pi}} \int_{\pi}^{2\pi} \sin(u)\,du = \frac{1}{2\sqrt{\pi}} \left(-\cos(u)\right)\Big|_{\pi}^{2\pi}$$

$$= \frac{1}{2\sqrt{\pi}}\left(-\cos(2\pi) + \cos(\pi)\right)$$

$$= \frac{1}{2\sqrt{\pi}}(-1-1) = \boxed{\frac{-1}{\sqrt{\pi}}}$$

6. (15 points)

(a) A parrot starts with a concentration of medicine in his bloodstream equal to 10 milligrams per liter (mg/L). Each day, the parrot uses up 45% of the medicine in his bloodstream. However, at the end of each day the vet gives him enough medication to increase the concentration of medicine in the bloodstream by 5 mg/L. Let $M_t =$ concentration of medicine on day $t$, and write down a discrete-time dynamical system, together with an initial condition, that describes this situation.

$$M_0 = 10$$

$$M_{t+1} = (1-0.45)M_t + 5$$

$$= 0.55 M_t + 5$$

(b) Let $H(t) =$ the height (in meters) of a tree at time $t$ (in years). Suppose that the tree grows at a rate $\frac{dH}{dt} = \frac{2t}{10+t^2}$) meters per year.

i. Use a definite integral to determine the total change in the height of the tree between times $t = 2$ and $t = 12$.

total change in H btwn $t=2$ & $t=12$ $= \int_2^{12} \frac{2t}{10+t^2}\,dt = \int_{14=u(2)}^{154=u(12)} \frac{du}{u} = \ln u \Big|_{14}^{154} = \ln(154) - \ln(14)$

$= \ln\left(\frac{154}{14}\right) = \ln(11)$

$\simeq \underline{2.400}$

$u \overset{set}{=} 10+t^2$

$du = 2t\,dt$

ii. Determine $H(t)$ if $H(0) = 1$. (That is, find a solution to the differential equation $\frac{dH}{dt} = \frac{2t}{10+t^2}$) with initial condition $H(0) = 1$.)

$H(t) = \int \frac{2t}{10+t^2}\,dt = \int \frac{du}{u} = \ln|u| + C$

$u = 10+t^2$

$du = 2t\,dt$

$= \underline{\ln(10+t^2) + C}$

$1 = H(0) = \ln(10+0) + C$

$\Rightarrow C = 1 - \ln(10)$

$\Rightarrow C \simeq -1.302;$

$$\boxed{H(t) = \ln(10+t^2) - 1.302}$$

7. (14 points)

(a) A population of striped goldfish obeys the discrete-time dynamical system

$$g_{t+1} = 1.3g_t.$$

(i) Write down the solution to this discrete-time dynamical system if $g_0 = 325$.
(ii) If $g_0 = 325$, at what time will the population reach size 1000?

(i) $g_t = 325(1.3)^t$

(ii) $1000 = 325(1.3)^t$

$\frac{1000}{325} = (1.3)^t$

$\ln\left(\frac{1000}{325}\right) = \ln\left(1.3^t\right) = t\ln(1.3);$

$t = \frac{\ln\left(\frac{1000}{325}\right)}{\ln(1.3)} \approx \underline{4.2838} \approx \underline{4.3}.$

(b) The density $\rho$ of a very thin rod (measured in grams/cm) varies according to

$$\rho(x) = \frac{3+6x}{1+x+x^2},$$

where $x$ marks a location along the rod, and $x = 0$ at one end of the rod. What is the total mass of the rod if it is 4 cm long? Give units in your answer.

mass $= \int_0^4 \rho(x)\,dx = \int_0^4 \frac{3+6x}{1+x+x^2}\,dx = 3\int_0^4 \frac{1+2x}{1+x+x^2}\,dx$

$u \overset{\text{set}}{=} 1+x+x^2$ $\quad u(0)=1$

$du = (1+2x)\,dx$ $\quad u(4) = 1+4+16 = 21$

$= 3\int_1^{21} \frac{du}{u} = 3(\ln 21 - \ln 1) = \boxed{3\ln(21) \text{ grams} \approx 9.1336 \text{ grams}}$

(c) Suppose that a population $b(t)$ of honey badgers satisfies the differential equation

$$\frac{db}{dt} = 1.3b(4-b).$$

i. Find all equilibria of the differential equation.

$0 \overset{\text{set}}{=} 1.3b(4-b) \implies b=0$, or $\underline{b=4}$

ii. Write down an initial condition for which the population will increase in time (at least initially).

$b(0) = \underline{\quad 2 \quad}.$

↑ any $b(0)$ such that $0 < b(0) < 4$.