MATH 155 Final Exam

Spring 2012

1. (14 points) Suppose that the population q_t of quails satisfies the discrete-time dynamical system

$$q_{t+1} = q_t(4 - q_t) - hq_t,$$

where $h > 0$ is a positive parameter, and q_t is measured in thousands of quails.

(a) Find all equilibria. For what values of h is there more than one nonnegative equilibrium?

$q_{t+1} = q_t(4-q_t) - hq_t$

$q^* = q^*(4-q^*) - hq^* = 4q^* - q^{*2} - hq^*$

$0 = 3q^* - hq^* - q^{*2} = q^*(3-h-q^*)$

$\rightarrow \boxed{q^*=0}$

or

$\boxed{q^* = 3-h}$

For $(0<)h < 3$, there is more than one nonnegative equilibrium.

(b) For each equilibrium, use the Stability Theorem/Criterion to determine the values of h for which that equilibrium is stable. Show clearly how you are using the Stability Theorem/Criterion.

updating function: $f(q) = q(4-q) - hq$

$= 4q - q^2 - hq$;

$f(q) = (4-h)q - q^2$.

$f'(q) = 4 - h - 2q$.

Stability Criterion: q^* is stable if $|f'(q^*)| < 1$.

$\underline{q^*=0}$: $|f'(0)| < 1$ if $|4-h-2(0)| < 1$; $|4-h| < 1$

$-1 < 4-h < 1$

$-5 < -h < -3$;

$q^*=0$ is stable if $\boxed{3 < h < 5}$

$\underline{q^* = 3-h}$: $|f'(3-h)| < 1$ if $|4-h-2(3-h)| < 1$; $|4-h-6+2h| < 1$

$|-2+h| < 1$

$-1 < -2+h < 1$

$q^* = 3-h$ is stable if .... $\rightarrow \boxed{1 < h < 3}$

2. (14 points) (a) Consider the function $f(x) = x^3 - \pi x + 3$. i) Find *all* critical points of $f(x)$. ii) Determine the global maximum and global minimum of $f(x)$ on the interval $[0, 3]$. Justify your answer and show your work clearly for full credit.

(i) $f'(x) = 3x^2 - \pi \overset{\text{set}}{=} 0. \rightarrow x^2 = \pi/3$, $x = \pm\sqrt{\pi/3} \approx \pm 1.023$

(ii) Possible global max's and min's occur at:

• Endpoints: $f(0) = 0^3 - \pi(0) + 3 = 3$

$f(3) = 3^3 - 3(\pi) + 3 = 30 - 3\pi \approx \boxed{20.575}$ Global Maximum at $(x, f(x)) = (3, \approx 20.575)$

and

• Critical Points in $[0,3]$

$f(+\sqrt{\pi/3}) = \boxed{0.8567}$: Global minimum at $(x, f(x)) \approx (\sqrt{\pi/3}, 0.8567)$.

(b) Suppose that the production of a pharmaceutical drug, D, depends on the population (measured in thousands) of some fungus p, in the following manner:

$$D(p) = 3p^2 e^{-0.001p}.$$

Find the *positive* critical point of the function $D(p)$, and use either the first or second derivative test to determine if there is either a local maximum or local minimum at that point. Show your work clearly for full credit.

$D'(p) = 3p^2 \frac{d}{dp}(e^{-0.001p}) + 3\frac{d}{dp}(p^2)e^{-0.001p}$

↑ Product Rule

$= 3p^2(-0.001 e^{-0.001p}) + 6pe^{-0.001p}$

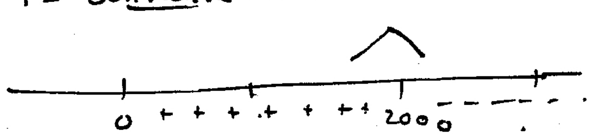
$= 3pe^{-0.001p}(2 - 0.001p)$

$D'(p) = 3pe^{-0.001p}(2 - 0.001p) \overset{\text{set}}{=} 0 \rightarrow p = 0,$ or $2 - 0.001p = 0;$

$p = \frac{2}{0.001} = 2000$

positive critical point at $p = 2000$

1st derivative test:



$D'(1000)$

$= \underbrace{3(1000)e^{-0.001(1000)}}_{>0} \underbrace{(2 - 0.001 \cdot 1000)}_{>0} > 0$

$D'(3000) = \underbrace{3(3000)e^{-0.001(3000)}}_{>0} \underbrace{(2 - 0.001 \cdot 3000)}_{<0} < 0$

There is a local maximum at $p = 2000$ according to the 1st derivative test.

3. (15 points) Evaluate the following definite and indefinite integrals. If necessary, use substitution. Show all of your work.

(a) $\int 4t^{\frac{3}{2}} - \frac{6}{t} + \pi \, dt$ $= 4\left(\frac{2}{5}\right)t^{5/2} - 6\ln|t| + \pi t + C$

(b) $\int 5\sin(x)e^{\cos(x)+\pi} \, dx$ $= -5\int e^u \, du = -5e^u + C$

$u \overset{set}{=} \cos(x) + \pi$

$du = -\sin(x)\,dx$

$= \boxed{-5e^{\cos(x)+\pi} + C}$

(c) $\int_0^1 \frac{4x}{x^2+2} \, dx$ $= \int_{u(0)=2}^{u(1)=3} \frac{2}{u} \, du = 2\ln|u| \Big|_2^3$

$u \overset{set}{=} x^2 + 2$

$du = 2x\,dx$

(So, $2\,du = 4x\,dx$)

$= 2\ln(3) - 2\ln(2)$

$= 2\ln(3/2)$

$(\simeq 0.8109\ldots)$

(d) Use integration by parts to evaluate $\int 2x \sin(7x) \, dx$.

$u = 2x \qquad dv = \sin(7x)dx$

$du = 2dx \qquad v = -\frac{1}{7}\cos(7x)$

$\int 2x \sin(7x) dx = \underbrace{-\frac{2x}{7}\cos(7x)}_{uv} \underbrace{- \int -\frac{1}{7}\cos(7x) \cdot 2dx}_{-\int v\,du}$

$= \boxed{-\frac{2}{7}x\cos(7x) + \frac{2}{7}\frac{1}{7}\sin(7x) + C}$

4. (14 points) Suppose that a bacterium is absorbing copper from its environment. At time $t = 0$, there is 0.6 mol of copper in the bacterium, and copper enters the bacterium at a rate of $\frac{1}{2+t^2} \frac{\text{mol}}{\text{min}}$

(a) Let $c(t)$ represent the amount (mol) of copper in the bacterium at time t (minutes). Write a pure-time differential equation and an initial condition for the situation described above.

$$\frac{dc}{dt} = \frac{1}{2+t^2}$$

$$c(0) = 0.6$$

(b) Apply Euler's Method with $\Delta t = 0.5$ to estimate the amount of copper in the bacterium at time $t = 1.5$. Show your work clearly using a table.
(Recall the formula $\hat{c}_{\text{next}} = \hat{c}_{\text{current}} + \frac{dc}{dt}\Delta t$, or $\hat{c}(t + \Delta t) = \hat{c}(t) + c'(t)\Delta t$).

t	$\hat{c}_{\text{current}}$	$\frac{dc}{dt} = \frac{1}{2+t^2}$	$\hat{c}_{\text{next}} = \hat{c}_{\text{current}} + \frac{dc}{dt}\Delta t$ $= \hat{c}_{\text{current}} + \frac{1}{2+t^2}(0.5)$
0	0.6	$\frac{1}{2} = 0.5$	$\hat{c}(0.5) = 0.6 + 0.5(0.5) = 0.85$
0.5	0.85	$0.\overline{44}$	$\hat{c}(1) = 0.85 + 0.\overline{44}(0.5) \cong 1.072$
1	1.072	$\frac{1}{3} = 0.\overline{3}$	$\hat{c}(1.5) = 1.072 + 0.\overline{3}(0.5) \simeq 1.2386... \simeq 1.239$
1.5	$\boxed{1.239}$		

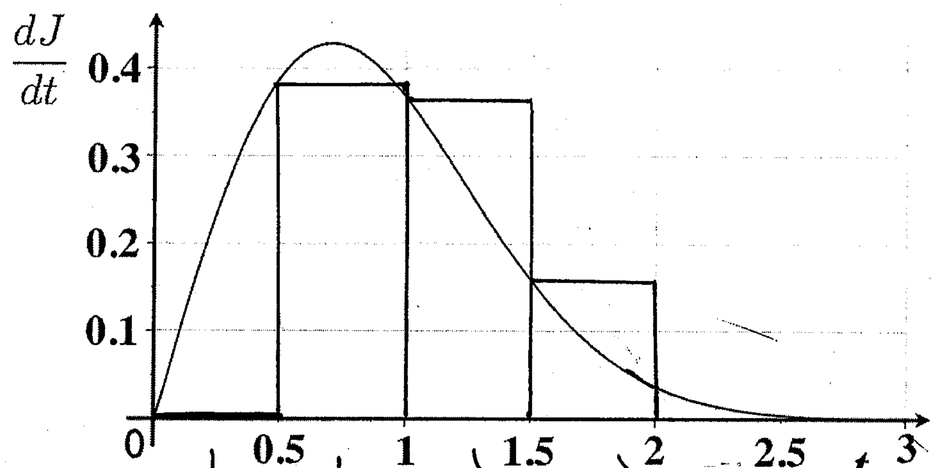
$$\boxed{\hat{c}(1.5) \simeq 1.239}$$

5. (14 points) Your pancreas produces insulin depending on the time since you ate green and gold ice cream. Assume that the rate of insulin production is

$$\frac{dJ}{dt} = te^{-t^2},$$

where time t is measured in hours and $J(t)$ is the amount of insulin (in moles) that the pancreas has produced t hours after you ate the ice cream.

(a) Estimate the total change in $J(t)$ between times $t = 0$ and $t = 2$ using a left-hand Riemann Sum with $\Delta t = 0.5$. Draw your rectangles or step functions on the graph of $\frac{dJ}{dt}$ below. Give your answer to three decimal places.



$$\text{LHRS} = 0(0.5) + (0.5)e^{-0.5^2}(0.5) + (1)e^{-1^2}(0.5) + (1.5)e^{-1.5^2}(0.5)$$

$$\approx 0.5(0 + 0.3894\ldots + 0.367879\ldots + 0.15809\ldots)$$

$$\approx 0.457689\ldots \approx \boxed{0.458}$$

(b) Find the exact area under the curve te^{-t^2} between times $t = 0$ and $t = 2$. What does this area represent biologically?

$$\text{exact area} = \int_0^2 te^{-t^2}\,dt = -\frac{1}{2}\int_{u(0)=0}^{u(2)=-4} e^u\,du = -\frac{1}{2}e^u\Big|_0^{-4}$$

$u \overset{\text{set}}{=} -t^2$

$du = -2t\,dt$

$$= -\frac{1}{2}\left(e^{-4} - e^0\right)$$

$$= \frac{1}{2}\left(1 - e^{-4}\right)$$

$$\boxed{\approx 0.491}$$

This area represents the total amount of insulin produced by the pancreas between times $t=0$ and $t=4$.

6. (15 points)

(a) To celebrate completion of Math 155, you plant an elm tree on the oval. The tree is only 260 cm tall when planted, but it grows 80 cm per year after being planted. Let h_t = the height of the tree (in cm) t years after being planted, and write down a discrete-time dynamical system, together with an initial condition, that describes this situation.

$$h_0 = 260$$
$$h_{t+1} = h_t + 80$$

(b) Let $L(t)$ = the length (in cm) of a fish at time t (in years). Suppose that the fish grows at a rate $\frac{dL}{dt} = 5.0e^{-0.2t}$.

i. Use a definite integral to determine the total change in length of the fish between times $t = 5$ and $t = 10$.

total change in length btwn $t=5$ and $t=10$ $= \int_5^{10} 5.0e^{-0.2t}\,dt = 5\int_5^{10} e^{-0.2t}\,dt$

$$= 5\cdot\frac{1}{-0.2}e^{-0.2t}\Big|_5^{10} = -25e^{-0.2(10)} - (-25e^{-0.2(5)})$$

↑ use w-substitution; ($w = -0.2t$)

$$= 25(e^{-1} - e^{-2}) \approx \boxed{5.81\text{cm}}$$

ii. Determine $L(t)$ if $L(0) = 2$. (That is, find a solution to the differential equation $\frac{dL}{dt} = 5.0e^{-0.2t}$ with initial condition $L(0) = 2$.)

$$L(t) = \int 5.0e^{-0.2t} = \frac{5}{-0.2}e^{-0.2t} + c = -25e^{-0.2t} + c$$

↑ From (i)

$$2 = L(0) = -25e^{-0.2(0)} + c;$$
$$2 = -25 + c;$$
$$c = 27;$$
$$\boxed{L(t) = -25e^{-0.2t} + 27}$$

7. (14 points)

(a) Use the tangent-line approximation of the function $f(x) = \sqrt{x}$ to approximate $\sqrt{16.1}$. Give your answer to four decimal places.

For $f(x) = \sqrt{x}$, $f(16 \approx 16.1) = \sqrt{16} = 4$.

The tangent line approximation to $f(x)$ at $x = 16$ is

$$\hat{f}_{16}(x) = f(16) + f'(16)(x-16)$$

$f'(x) = \frac{1}{2}x^{-1/2}$

$$= \sqrt{16} + \frac{1}{2}(16)^{-1/2}(x-16)$$

$$= 4 + \frac{1}{2}\left(\frac{1}{4}\right)(x-16) = 4 + \frac{1}{8}(x-16).$$

$$\sqrt{16.1} = f(16.1) \simeq \hat{f}(16.1) = 4 + \frac{1}{8}(16.1-16) = 4 + \frac{1}{8}(0.1) = \boxed{4.0125}$$

(b) The density ρ of a very thin rod (measured in grams/cm) varies according to

$$\rho(x) = 1 + \frac{x}{1+x^2},$$

where x marks a location along the rod, and $x = 0$ at one end of the rod. What is the total mass of the rod if it is 10 cm long? Give units in your answer.

$$\text{total mass} = \int_0^{10} \rho(x)\,dx = \int_0^{10} 1 + \frac{x}{1+x^2}\,dx$$

$$= \int_0^{10} 1\,dx + \int_0^{10} \frac{x}{1+x^2}\,dx = x\Big|_0^{10} + \int_0^{10} \frac{x}{1+x^2}\,dx = 10 - 0 + \int_{w(0)=1}^{w(10)=101} \frac{1}{2}\frac{1}{w}\,d$$

$w \overset{set}{=} 1 + x^2$

$dw = 2x\,dx$

$$= 10 + \frac{1}{2}\ln|w|\Big|_1^{101} = 10 + \frac{1}{2}\big(\ln(101) - \underbrace{\ln(1)}_{0}\big) = 10 + \frac{1}{2}\ln(101)$$

$$\cong \boxed{12.3 \text{ grams}}$$

(c) Suppose that the population $w(t)$ of warblers satisfies the differential equation

$$\frac{dw}{dt} = 2w(8-w).$$

i. Find all equilibria of the differential equation.

$0 \overset{set}{=} 2w(8-w)$

$$\boxed{\begin{array}{l} w = 0 \\ w = 8 \end{array}}$$

ii. Write down an initial condition for which the population will increase in time (at least initially).

$w(0) =$ __4__.

$(0 < w(0) < 8)$