Math 155 **Final Exam** **Spring 2006**

NAME: ______________________________

SECTION: _______________

INSTRUCTOR: ______________________________

Instructions: The exam is closed book and closed notes. You may use a calculator, but be sure to show your work on each problem for full credit. Work that is crossed out or erased will not be graded. All work that is not crossed out will be graded. Turn in any scratch paper that you use during the exam. You will have two hours to work on the exam.

Problem	Points	Score
1	16	
2	15	
3	14	
4	16	
5	12	
6	12	
7	15	
Total	100	

1. (16 pts) Evaluate the following definite and indefinite integrals. If necessary, use u-substitution.

a) $\int 2t\cos(t^2-\pi)dt.$

$w = t^2 - \pi$

$\frac{dw}{dt} = 2t$

$\frac{dw}{2t} = dt$

$\int 2t\cos(w)\left(\frac{dw}{2t}\right)$

$= \int \cos w \, dw$

$= \boxed{\sin(t^2-\pi)+c}$

b) $\int \left(2x^{-3/2}+10^3-x^4\right)dx.$

$= \frac{2}{-1/2}x^{-1/2} + 10^3x - \frac{x^5}{5} + c$

$= \boxed{-4x^{-1/2} + 10^3x - \frac{1}{5}x^5 + c}$

c) $\int_1^e \left(\frac{1}{t}-t^{-2}\right)dt.$ $= \int_1^e \left(t^{-1}-t^{-2}\right)dt$

$= \ln|t| + t^{-1}\Big|_1^e$

$= \ln(e) + e^{-1} - \ln(1) - 1^{-1} = \boxed{\frac{1}{e}}$

d) $\int_0^1 \frac{2t}{t^2+4}dt.$

$w = t^2+4$

$\frac{dw}{dt} = 2t$

$\frac{dw}{2t} = dt$

$\int \frac{2t}{w}\left(\frac{dw}{2t}\right)$

$= \int \frac{1}{w}dw$ $= \ln(|t^2+4|)\Big|_0^1$

$= \ln(5) - \ln(4)$

$\boxed{\approx .2231}$

2. a.) (7 pts) Evaluate the improper integral. If necessary, use u-substitution.

$\int_0^{\infty} e^{-7t}dt.$ $= \lim_{b\to\infty} \int_0^b e^{-7t}dt$

$= \lim_{b\to\infty} -\frac{1}{7}e^{-7t}\Big|_0^b$

$= \lim_{b\to\infty} \left(-\frac{1}{7}e^{-7b} + \frac{1}{7}e^{-7(0)}\right)$

$= \boxed{\frac{1}{7}}$

b) (8 pts) Use integration by parts to evaluate the definite integral.

$$\int_0^1 (2t-1)e^{-3t}dt$$

$u = 2t-1$ $\qquad$ $v = e^{-3t}$

$du = 2$ $\qquad$ $dv = -\frac{1}{3}e^{-3t}$

$= (2t-1)\left(-\frac{1}{3}e^{-3t}\right) - \int 2\left(-\frac{1}{3}e^{-3t}\right)dt$

$= -\frac{2}{3}te^{-3t} + \frac{1}{3}e^{-3t} + \frac{2}{3}\left(-\frac{1}{3}e^{-3t}\right)\Big|_0^1$

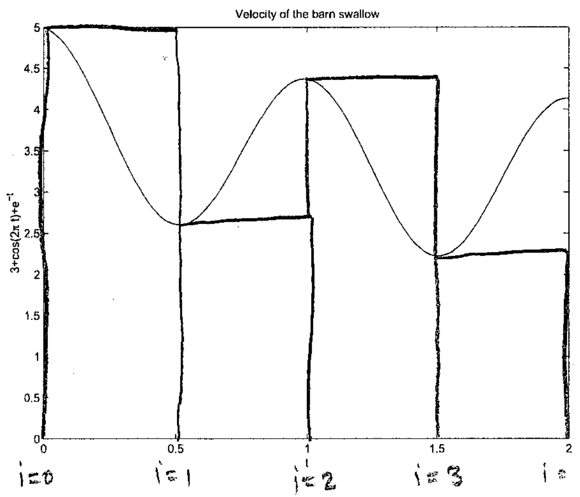
$= -\frac{2}{3}e^{-3} + \frac{1}{3}e^{-3} - \frac{2}{9}e^{-3} + 0 - \frac{1}{3} + \frac{2}{9}$

$= \boxed{-\frac{5}{9}e^{-3} - \frac{1}{9}}$

3. (14 pts) On a calm day, an unladen barn swallow flies with velocity (km/hr)

$$\frac{dP}{dt} = 3 + \cos(2\pi t) + e^{-t}$$

a) Estimate the total distance traveled between $t = 0$ and $t = 2$ using a Left-hand Riemann sum with $\Delta t = 0.5$. Draw your horizontal lines or rectangles in the figure below.



$$I_l = \sum_{i=0}^{3} \left(3 + \cos(2\pi t_i) + e^{-t_i}\right)(0.5)$$

$$\approx 2.5 + 1.3033 + 2.1839 + 1.1116$$

$$\approx \boxed{7.0988}$$

b) Find the area under the curve of $\frac{dP}{dt}$ between $t = 0$ and $t = 2$. What does this quantity represent for the barn swallow?

$$\int_0^2 \left(3 + \cos(2\pi t) + e^{-t}\right) dt$$

$$= 3t + \frac{1}{2\pi}\sin(2\pi t) - e^{-t}\Big|_0^2$$

$$= 6 + \frac{1}{2\pi}\sin(4\pi) - e^{-2} - 0 - \frac{1}{2\pi}\sin(0) + e^{0}$$

$$= \boxed{7 - e^{-2}}$$

distance traveled

4. (12 pts) a) Use Newton's Method to find the nonzero equilibrium of the updating function

$$N_{t+1} = 3N_t e^{-N_t} - N_t.$$

Take one step from an initial guess of 1. The formula for Newton's method to solve $f(x) = 0$ is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

$$N^* = 3N^* e^{-N^*} - N^*$$

$$0 = 3N^* e^{-N^*} - 2N^* = f(N)$$

$$f'(N) = 3N(e^{-N})(-1) + e^{-N}(3) - 2$$

$$= -3Ne^{-N} + 3e^{-N} - 2$$

$$N_0 = 1$$

$$N_1 = 1 - \frac{3e^{-1} - 2}{-3e^{-1} + 3e^{-1} - 2} \approx \boxed{.5518}$$

b) Now find the exact values of the equilbria algebraically.

$$0 = 3N^* e^{-N^*} - 2N^*$$

$$0 = N^*(3e^{-N^*} - 2)$$

$$\boxed{0 = N^*} \quad \text{and} \quad 3e^{-N^*} = 2$$

$$e^{-N^*} = 2/3$$

$$\boxed{N^* = -\ln(2/3) \approx .4055}$$

c) Use the stability test to determine the stability of the nonzero equilibrium point. State the stability test or show clearly how you are using it. No partial credit will be given for solutions that do not use the stability test.

$$g(N) = 3Ne^{-N} - N$$

$$g'(N) = 3N(-e^{-N}) + e^{-N}(3) - 1$$

$$N^* = -\ln(2/3)$$

$$g'(N^*) = -3\ln(2/3)(-e^{\ln(2/3)}) + e^{\ln(2/3)}(3) - 1$$

$$= 3\ln(2/3)(2/3) + 3(2/3) - 1$$

$$= 2\ln(2/3) + 2 - 1$$

$$\approx |.1891| < 1 \quad \text{stable}$$

5. (12 pts) Suppose that the production of a pharmaceutical agent, Q, depends on the population of some bacteria, B, in the following manner:

$$Q(B) = 2Be^{-0.002B}$$

The units on B are thousands of bacteria.
a) For what population level B is the production Q is at a maximum? Justify your answer with either the first or second derivative test.

$$Q'(B) = 2B\,e^{(-0.002B)}(-0.002) + e^{-0.002B}(2) \overset{\text{set}}{=} 0$$

$$2e^{-0.002B}\left(B(-0.002) + 1\right) = 0$$

$$-0.002B = -1$$

$$B = 500$$

$Q'(B)$ + | −

500

$Q'(0) > 0$

$Q'(1000) < 0$

so it is a maximum.

b) What is $\lim_{B\to\infty} Q(B)$? Hint: $Q(B) = \frac{2B}{e^{.002B}}$.
Show all of your work. If you use L'Hospital's Rule, justify why it can be applied each time you use it. If you use knowledge of leading behaviors, justify your work.

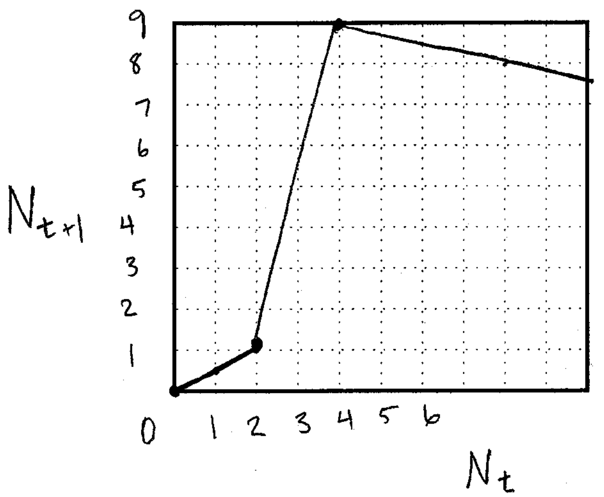
$$\lim_{B\to\infty} \frac{2B}{e^{0.002B}} \quad \text{form } \frac{\infty}{\infty} \quad \text{L'Hôpital's}$$

$$\lim_{B\to\infty} \frac{2}{0.002\,e^{0.002B}} = \boxed{0}$$

6. (12pts) Let N_t represent the difference between the sodium concentration inside and outside of a cell at some time t in seconds. The updating function is

$$N_{t+1} = \begin{cases} 0.5N_t, & \text{if} \quad N_t \leq 2 \\ 4N_t - 7, & \text{if} \quad 2 < N_t \leq 4 \\ -.25N_t + 10, & \text{if} \quad 4 < N_t \end{cases}$$

a) Accurately graph the updating function. Is the function continuous? Why or why not?



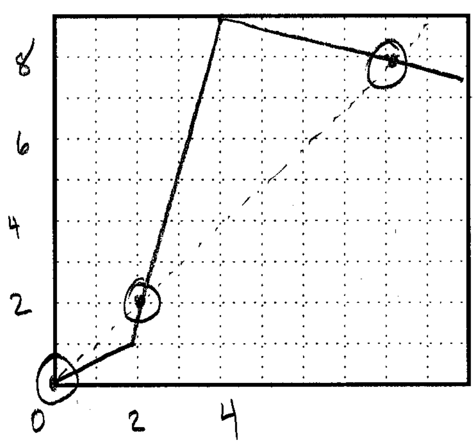
Yes, it is continuous because

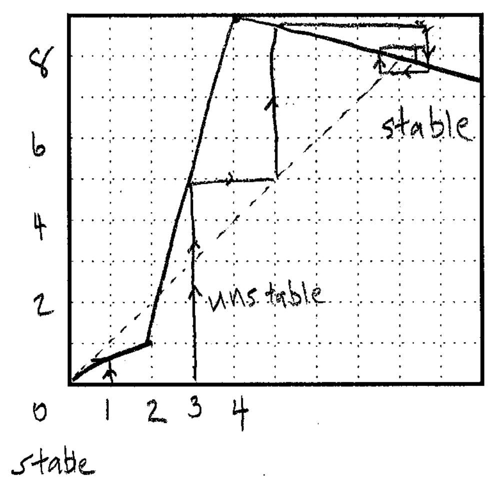
$$\lim_{N_t \to 2^+} N_{t+1} = \lim_{N_t \to 2^-} N_{t+1} = N_{t+1}(2)$$

and

$$\lim_{N_t \to 4^+} N_{t+1} = \lim_{N_t \to 4^-} N_{t+1} = N_{t+1}(4)$$

b) Resketch your graph **twice** below and include the diagonal $N_{t+1} = N_t$.
On the left plot: Circle all equilibrium points.
On the right plot: Draw a cobweb diagram starting from $N_0 = 1$ and $N_0 = 3$. Label each equilibrium as stable or unstable. Your cobweb diagram is sufficient justification for your answer.





7. (12pts) Let $R(t)$ represent the rate function for the production of amino acids in a cell:

$$R(t) = \frac{90}{27 + t^2}$$

a) Find the derivative $R'(t)$ of the rate function. What are the critical points? Where is the function increasing and decreasing?

$$R'(t) = \frac{(27+t^2)(0) - 90(2t)}{(27+t^2)^2} = \frac{-180t}{(27+t^2)^2} \overset{\text{set}}{=} 0$$

$$-180t = 0$$
$$t = 0$$

$+$ | $-$

0

$R'(-1) > 0$ increasing $(-\infty, 0)$

$R'(1) < 0$ decreasing $(0, \infty)$

b) Find $R''(t)$. You do not need to simplify your answer.

$$R''(t) = \frac{(27+t^2)^2(-180) - (-180t)(2(27+t^2)(2t))}{(27+t^2)^4}$$

c) Find $\lim\limits_{t\to\infty} R(t)$. Justify your answer.

$$\lim_{t\to\infty} \frac{90}{27+t^2} = \boxed{0}$$