Math 155 Final Exam
Fall 2012

1. (14 points) Evaluate the following definite and indefinite integrals. If necessary, use substitution. Show all of your work.

(a) $\int \frac{5x^4+2x^2}{x^3} + \pi\, dx$ $= \int\left(\frac{5x^4}{x^3} + \frac{2x^2}{x^3} + \pi\right)dx = \int(5x + 2x^{-1} + \pi)\,dx$

$= \frac{5}{2}x^2 + 2\ln|x| + \pi x + C$

(b) $\int t^3 \sin(t^4+7)dt$ $= \int \frac{1}{4}\sin(\omega)\,d\omega = -\frac{1}{4}\cos(\omega) + C = -\frac{1}{4}\cos(t^4+7) + C$

$\omega \overset{set}{=} t^4+7$

$d\omega = 4t^3dt;\ \frac{1}{4}d\omega = t^3dt$

(c) $\int_0^1 \frac{3}{(2x+1)^3}dx$ $= \int_1^3 \frac{3}{2}\frac{1}{\omega^3}d\omega = \frac{3}{2}\frac{\omega^{-2}}{(-2)}\Big|_1^3 = -\frac{3}{4}\frac{1}{\omega^2}\Big|_1^3 = -\frac{3}{4}\left(\frac{1}{9} - \frac{1}{1}\right)$

$\omega \overset{set}{=} 2x+1$ $\quad \omega(1) = 3$

$d\omega = 2dx;\ \frac{1}{2}d\omega = dx$ $\quad \omega(0) = 1$

$= -3/4(-8/9) = \boxed{2/3}$

(d) Use integration by parts to evaluate $\int 5xe^{4x}dx$.

Let $u = 5x$, $dv = e^{4x}dx$

$du = 5dx$, $v = \frac{1}{4}e^{4x}$

$\int 5xe^{4x}dx = uv - \int v\,du = 5x\left(\frac{1}{4}e^{4x}\right) - \int \frac{1}{4}e^{4x}(5dx)$

$= \frac{5}{4}xe^{4x} - \frac{5}{4}\left(\frac{1}{4}e^{4x}\right) + C$

$= \frac{5}{4}xe^{4x} - \frac{5}{16}e^{4x} + C$

2. (14 points) Consider the discrete-time dynamical system

$$N_{t+1} = 2N_t(1 - N_t) - hN_t$$

describing a population of shrimp being harvested at rate $h \geq 0$.

(a) Find the nonzero equilibrium population N^* as a function of h. For what values of h is there a positive equilibrium?

$$N^* = 2N^*(1-N^*) - hN^*;$$
$$N^* = 2N^* - 2N^{*2} - hN^*$$
$$0 = N^* - 2N^{*2} - hN^*$$
$$= N^*(1-h-2N^*)$$

$\Rightarrow N^* = 0$, or

$1-h-2N^* = 0$; $\boxed{N^* = \frac{1-h}{2}}$

$\boxed{N^* = \frac{1-h}{2} \text{ is } > 0 \text{ if } \underline{h<1} \quad (0 \leq h < 1)}$

(b) The equilibrium harvest is given by $P(h) = hN^*$, where N^* is the equilibrium you found in part (a). Find the value of h that maximizes $P(h)$. Use the first or second derivative test to justify that this value of h gives a local maximum.

$$P(h) = hN^* = h\left(\frac{1-h}{2}\right) = \frac{1}{2}(h-h^2)$$
$$P'(h) = \frac{1}{2} - h$$

critical point:

$$P'(h) = \frac{1}{2} - h \overset{set}{=} 0 \Rightarrow \underline{h = 1/2}.$$

1st derivative test:

$+ + + +_0 + + + _{1/2} - - - - - - -$ $\Rightarrow$ P has a $\underline{\text{local max}}$ at $h = 1/2$

$P'(0) = 1/2 > 0$ $\quad$ $P'(1) = -1/2 < 0$

2nd derivative test:

$P''(h) = 1/2$

$\Rightarrow P''(1/2) = 1/2 > 0 \Rightarrow$ P has a local $\underline{\text{max at } h = 1/2}$.

(c) Use the Stability Test/Criterion to determine if the equilibrium you found in (a) is stable if $h = \frac{1}{2}$. $N_{t+1} = 2N_t(1-N_t) - hN_t = (2-h)N_t - 2N_t^2 = f(N_t)$. $f'(N_t) = 2-h-4N_t$

$N^* = \frac{1-h}{2}$ is stable if $|f'(\frac{1-h}{2})| < 1$

$$\left|f'\left(\tfrac{1-h}{2}\right)\right| = \left|2-h-4\left(\tfrac{1-h}{2}\right)\right| = \left|2-h-2(1-h)\right| = |h|, \text{ which is } < 1 \text{ if}$$

$\underline{-1 < h < 1}$

($h \geq 0$, so actually $\underline{0 \leq h < 1}$)

$\Rightarrow \boxed{N^* = \frac{1-h}{2} \text{ is stable if } 0 \leq h < 1}$

3. (15 points) Consider the function $f(x) = -x^3 + 3x^2 + 9x$ on the interval $[-3.5, 4]$.

(a) Calculate $f'(x)$, and use this to find all the critical points of $f(x)$.

$f'(x) = -3x^2 + 6x + 9 \overset{set}{=} 0$

$\Rightarrow x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0 \Rightarrow$ critical points:

$x = -1$

$x = 3$

(b) Calculate $f''(x)$, and use this to find regions where $f(x)$ is concave up or concave down.

$f''(x) = -6x + 6$

$-6x + 6 \overset{set}{=} 0 \rightarrow x = 1$

$f''(0) = 6 > 0$ $f''(1) = 0$ $f''(2) = -6 < 0$

0 $x = 1$ 2

$f''(x) > 0$ for $x < 1$: concave up for $x < 1$

$f''(x) < 0$ for $x > 1$: concave down for $x > 1$.

(c) For each critical point, determine if $f(x)$ has a local maximum or a local minimum there. Justify your answer using the *first derivative test.*

$x = -1$:

$f'(-2) = -15 < 0$ $f'(-1) = 0$ $f'(0) = 9 > 0$ $f'(3) = 0$ $f'(4) = -15 < 0$

-2 $x = -1$ 0 3 4

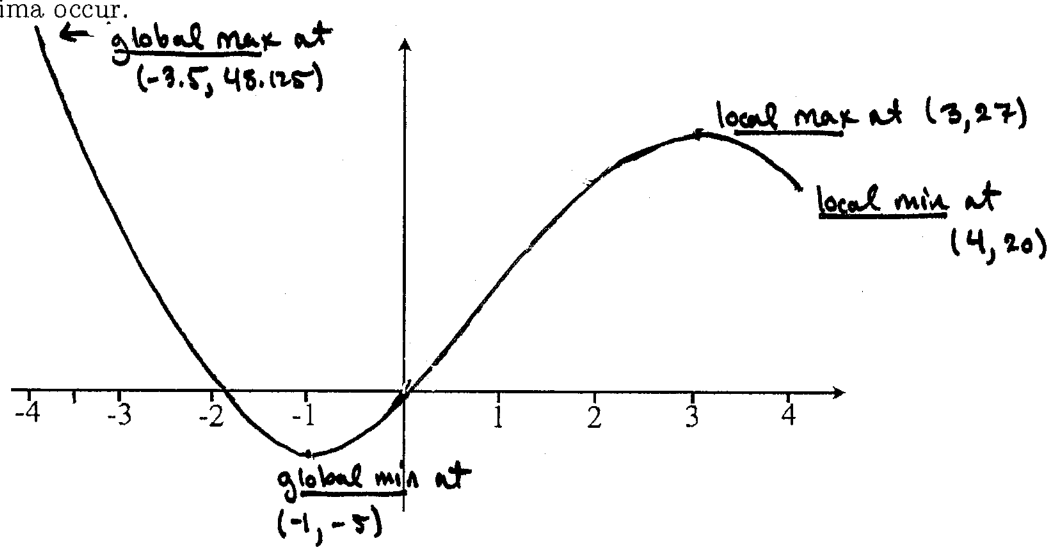
- - - - - - - - + + + + + + + + + + + + - - - - - - - - -

dec incr incr decr

f has a local minimum at $x = -1$.

f has a local maximum at $x = 3$.

(d) Use the information found above to sketch a graph of the function $f(x)$ on the interval $[-3.5, 4]$. Indicate where any local maxima, local minima, global maxima, or global minima occur.



4. (14 points) Let $P(t)$ be the amount (in moles) of a chemical being formed in a reaction. Suppose that the rate at which the chemical is being formed is given by

$$\frac{dP}{dt} = \frac{2t}{t^2+5} \text{ moles/sec.}$$

(a) What is $\lim_{t\to\infty} \frac{dP}{dt}$?

$$\lim_{t\to\infty} \frac{2t}{t^2+5} \underset{\text{L.H.}}{=} \lim_{t\to\infty} \frac{2t}{t^2} = \lim_{t\to\infty} \frac{2}{t} = \boxed{0}.$$

(b) If $P(0) = 5$, what is $P(t)$? (Solve the initial-value problem for $P(t)$).

$$P(t) = \int \frac{dP}{dt}\,dt = \int \frac{2t}{t^2+5}\,dt = \int \frac{1}{w}\,dw = \ln|w| + C$$

$w \overset{\text{set}}{=} t^2+5$

$dw = 2t\,dt$

$$= \ln|t^2+5| + C$$

If $P(0)=5$, then

$5 = P(0) = \ln|0^2+5| + C$

$\Rightarrow C = 5 - \ln(5)$ $\Rightarrow \boxed{P(t) = \ln|t^2+5| + (5-\ln(5))}$

(c) Find the average rate at which product is being formed (that is, the average value of $\frac{dP}{dt} = \frac{2t}{t^2+5}$) between times $t = 0$ and $t = 2$.

Average rate of change on $[0,2]$ $= \frac{1}{2-0}\int_0^2 \frac{2t}{t^2+5}\,dt \underset{\text{from part b}}{=} \frac{1}{2}\cdot \ln|t^2+5|\Big|_0^2$

$$= \frac{1}{2}\ln(9) - \frac{1}{2}\ln(5)$$

$$= \boxed{\frac{1}{2}\ln\left(\frac{9}{5}\right)} \approx \underline{0.2938}$$

(d) Use a definite integral to find the total change in the amount of product between times $t = 1$ and $t = 5$.

ΔP between $t=1$ and $t=5$ $= \int_1^5 \frac{dP}{dt}\,dt = \int_1^5 \frac{2t}{t^2+5}\,dt \underset{\text{part b}}{=} \ln(t^2+5)\Big|_1^5$

$$= \ln(5^2+5) - \ln(1+5) = \ln\left(\frac{30}{6}\right) = \boxed{\ln(5)}$$

$$\approx \underline{1.6094}$$

5. (14 points) Suppose that a cell is absorbing a certain drug from its environment. At time $t = 0$, there is 10 mol of the drug in the cell, and the drug enters the cell at a rate of $1 + \sin(t^2)$ mol/min.

(a) Let $c(t)$ represent the amount (mol) of drug in the cell at time t (in minutes). Write a pure-time differential equation and an initial condition for the situation described above.

$$\frac{dc}{dt} = 1 + \sin(t^2)$$

$$c(0) = 10$$

(b) Apply Euler's Method with $\Delta t = 0.5$ to estimate the amount of drug in the cell at time $t = 1.5$. Show your work clearly using a table.
(Recall the formula $c_{\text{next}} = c_{\text{current}} + \frac{dc}{dt}\Delta t$, or $\hat{c}(t + \Delta t) = \hat{c}(t) + c'(t)\Delta t$.)

t	$\hat{c}_{current}$	$\frac{dc}{dt} = 1 + \sin(t^2)$	$\hat{c}_{next} = \hat{c}_{current} + 0.5\overbrace{(1+\sin(t^2))}^{\frac{dc}{dt}}$
0	10	1	$10 + 0.5 \cdot 1 = 10.5$
0.5	10.5	1.2474	$10.5 + 0.5(1.2474) = 11.1237$
1	11.1237	1.8415	$11.1237 + 0.5(1.8415) = 12.04445$
1.5	12.04445		

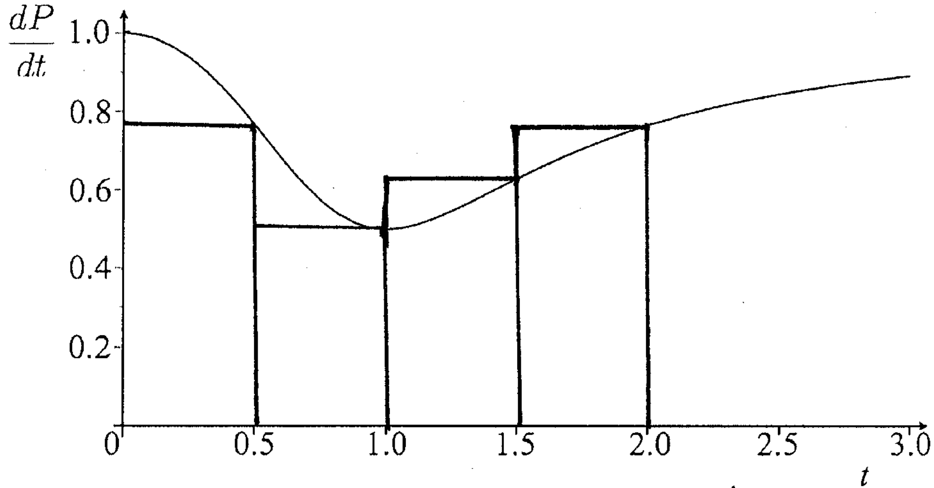
$$\boxed{\hat{c}(1.5) = 12.044}$$

6. (14 points)

(a) A flying big brown bat slows down a bit to catch a fly, and then increases its speed again as it flies on. Denoting the position (in meters) of the bat at time t (in seconds) by $P(t)$, suppose that the bat's velocity is given by

$$\frac{dP}{dt} = 1 - \frac{t^2}{1+t^4}.$$

Estimate the total change in $P(t)$ between times $t = 0$ and $t = 2$ using a right-hand Riemann Sum with $\Delta t = 0.5$. Draw your rectangles or step functions on the graph below.



ΔP between $t=0$ and $t=2$ $\approx \Delta t\left(\frac{dP}{dt}(0.5) + \frac{dP}{dt}(1) + \frac{dP}{dt}(1.5) + \frac{dP}{dt}(2.0)\right)$

↑ Right-Hand Riemann Sum

$$= 0.5\left(1 - \frac{0.5^2}{1+0.5^4} + 1 - \frac{1^2}{1+1^2} + 1 - \frac{1.5^2}{1+1.5^2} + 1 - \frac{2^2}{1+2^2}\right)$$

$$\approx \underline{1.3291}$$

(b) The density of a very thin rod varies according to $\rho(x) = \frac{1}{10}xe^{2x^2}$ in grams/cm, where x marks a location along the rod and $x = 0$ at one end of the rod. What is the total mass of the rod if it is 2 cm long?

$$\text{total mass} = \int_0^2 \rho(x)\,dx = \int_0^2 \frac{1}{10} x e^{2x^2}\,dx = \int_{w(0)=0}^{w(2)=8} \frac{1}{10}\cdot\frac{1}{4} e^{w}\,dw$$

$w \overset{\text{set}}{=} 2x^2$

$dw = 4x\,dx$; $\frac{1}{4}dw = x\,dx$

$$= \frac{1}{40} e^{w}\Big|_0^8 = \boxed{\frac{1}{40}\left(e^8 - e^0\right) \approx 74.5 \text{ grams}}$$

7. (15 points)

(a) A population of green, slimy algae obeys the discrete-time dynamical system

$$a_{t+1} = 1.8a_t.$$

(i) Write down the solution of this discrete-time dynamical system if $a_0 = 500$.
(ii) If $a_0 = 500$, at what time does the population reach size 1000?

(i) $a_t = 500 \cdot 1.8^t$

(ii) $1000 \overset{\text{set}}{=} 500 \cdot 1.8^{t_d}$

$2 = 1.8^{t_d}$

$\ln(2) = t_d \ln(1.8)$

$$\boxed{t_d = \frac{\ln(2)}{\ln(1.8)} \approx 1.179}$$

(b) Use a tangent-line approximation of the function $f(x) = \sqrt{x}$ to approximate $\sqrt{3.9}$. Give your answer to 3 decimal places.

tangent-line approximation $\hat{f}(x)$ to $f(x) = \sqrt{x}$ at $x = 4$:

$\hat{f}(x) - f(4) = f'(4)(x-4)$; $\quad (f'(x) = \frac{1}{2}x^{-1/2})$

$\hat{f}(x) - 2 = \frac{1}{2}(4)^{-1/2}(x-4) = \frac{1}{2} \cdot \frac{1}{2}(x-4)$;

$\hat{f}(x) = \frac{1}{4}(x-4) + 2$

Approximation of $\sqrt{3.9} = \hat{f}(3.9) = \frac{1}{4}(3.9-4) + 2 = -0.025 + 2$

$= \boxed{1.975}$

(c) Suppose that the population $k(t)$ of green kingfishers (measured in thousands) satisfies the differential equation

$$\frac{dk}{dt} = 3.1k(2.2 - k).$$

i) Find all equilibria of the differential equation.

$0 \overset{\text{set}}{=} \frac{dk}{dt} = 3.1k(2.2 - k) \Rightarrow \boxed{k = 0, \text{ or } k = 2.2}$

ii) Write down an initial condition for which the population will increase with time.

$k(0) =$ 1.69 $\in (0, 2.2)$.

any $k(0)$ such that $0 < k(0) < 2.2$ works.