Math 155 **Final Exam** **Fall 2009**

NAME: ______________________________

SECTION: ____________________ TIME: ____________________

INSTRUCTOR: ______________________________

Instructions: The exam is closed book and closed notes. You may use an approved calculator, but be sure to show your work on each problem for full credit. Work that is crossed out or erased will not be graded. Turn in any scratch paper that you use during the exam. You will have two hours to work on the exam.

Problem	Points	Score
1	14	
2	13	
3	13	
4	15	
5	8	
6	10	
7	13	
8	14	
Total	100	

CONFIDENTIALITY PLEDGE

I agree that I will not share any information, either specific or general, about the problems on this examination with any other person until the exams have been returned to us in class.

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(Signature)

1. (14 points) Evaluate the following definite and indefinite integrals. If necessary, use substitution. Show all of your work.

(a) $\int 2x^4 - \frac{3}{x}\,dx$

$\boxed{\frac{2}{5}x^5 - 3\ln(|x|) + c}$

(b) $\int t(t^2+1)^{-2}\,dt$

$w = t^2+1$

$\frac{dw}{dt} = 2t$

$\frac{dw}{2t} = dt$

$\int t\, w^{-2}\left(\frac{dw}{2t}\right)$

$= \frac{1}{2}\int w^{-2}\,dw$

$= \frac{1}{2}(-1)w^{-1} + c$

$= \boxed{-\frac{1}{2}(t^2+1)^{-1} + c}$

(c) $\int_0^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$w = \sqrt{x}$

$\frac{dw}{dx} = \frac{1}{2\sqrt{x}}$

$2\sqrt{x}\,dw = dx$

$\int \frac{e^w}{\sqrt{x}}(2\sqrt{x}\,dw)$

$= 2\int e^w\,dw$

$= 2e^{\sqrt{x}}\Big|_0^4$

$= \boxed{2e^2 - 2e^0 \approx 12.78}$

(d) Use integration by parts to evaluate $\int 2x\cos(5x)\,dx$.

$u = 2x$

$du = 2$

$dv = \cos(5x)$

$v = \frac{1}{5}\sin(5x)$

$\int \cos(5x)\,dw$

$w = 5x$

$\frac{dw}{dx} = 5$

$\frac{dw}{5} = dx$

$\int \cos(w)\,\frac{dw}{5}$

$= \frac{1}{5}\int \cos(w)\,dw$

$= \frac{1}{5}\sin(5x)$

$2x\left(\frac{1}{5}\sin(5x)\right) - \int 2\left(\frac{1}{5}\sin(5x)\right)dx$

$\frac{2}{5}x\sin(5x) - \frac{2}{5}\int \sin(5x)\,dx$

$w = 5x$

$\frac{dw}{dx} = 5$

$\frac{dw}{5} = dx$

$\int \sin(w)\left(\frac{dw}{5}\right)$

$= \frac{1}{5}\int \sin(w)\,dw$

$= -\frac{1}{5}\cos(5x)$

$\frac{2}{5}x\sin(5x) - \frac{2}{5}\left(-\frac{1}{5}\right)\cos(5x) + c = \boxed{\frac{2}{5}x\sin(5x) + \frac{2}{25}\cos(5x) + c}$

2. (13 points) Suppose that a cell starts at a volume of 400 μm^3 at time $t = 0$ and gains volume at a rate of $\frac{10}{1+t^2} \frac{\mu\text{m}^3}{\text{sec}}$.

(a) Let $V(t)$ represent the volume of the cell at time t. Write a pure-time differential equation and an initial condition for the situation described above.

$$\frac{dV}{dt} = \frac{10}{1+t^2}$$

$$V(0) = 400$$

(b) Apply Euler's Method with $\Delta t = 0.5$ to estimate the volume at time $t = 1.5$.

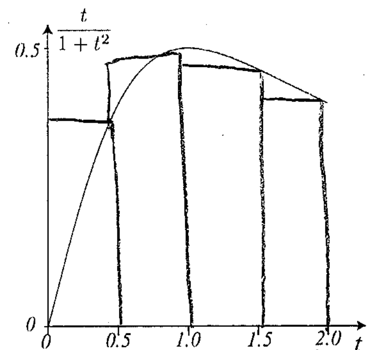
t	$\hat{V}_{current}$	$\frac{dV}{dt} = \frac{10}{1+t^2}$	$\hat{V}_{next} = \hat{V}_{current} + \frac{dV}{dt}(0.5)$
0	400	10	$400 + 10(.5) = 405$
0.5	405	8	$405 + 8(.5) = 409$
1	409	5	$409 + 5(.5) = 411.5$
1.5	411.5		

$\boxed{411.5 \, \mu m^3}$

3. (13 points) Let $c(t)$ be the amount (mol) of a drug in a bacterium at time t (minutes). The bacterium absorbs the drug from its environment, so that

$$\frac{dc}{dt} = \frac{t}{1+t^2}.$$

(a) Estimate the total change in $c(t)$ between times $t = 0$ and $t = 2$ using a right-hand Riemann sum with $\Delta t = 0.5$. Draw your rectangles or step function on the figure below.



$t_0 \quad t_1 \quad t_2 \quad t_3 \quad t_4$

$$I_r = \sum_{i=1}^{4} \frac{t_i}{1+t_i^2}(0.5) = \frac{.5}{1+.5^2}(0.5) + \frac{1}{1+1^2}(0.5) + \frac{1.5}{1+1.5^2}(0.5) + \frac{2}{1+2^2}(0.5)$$

$$\approx .2 + .25 + .2308 + .2$$

$$\approx \boxed{.8808 \text{ mol}}$$

(b) Find the *exact* area under the curve $\frac{dc}{dt} = \frac{t}{1+t^2}$ between $t = 0$ and $t = 2$. What does this quantity represent for the bacterium and the drug?

$$\int_0^2 \frac{t}{1+t^2}\,dt$$

$w = 1 + t^2$

$\frac{dw}{dt} = 2t$

$\frac{dw}{2t} = dt$

$$\int \frac{t}{w}\left(\frac{dw}{2t}\right)$$

$$= \frac{1}{2}\int \frac{1}{w}\,dw$$

$$= \frac{1}{2}\ln(|1+t^2|)\Big|_0^2$$

$$= \frac{1}{2}\ln(1+4) - \frac{1}{2}\ln(1)$$

$$\approx \boxed{.8047 \text{ mol}}$$

total amount of drug absorbed

4. (15 points)

(a) The population y_t of a yeast colony obeys the discrete-time dynamical system

$$y_{t+1} = 1.5 y_t.$$

Find the solution of this discrete-time dynamical system if $y_0 = 2.0 \times 10^3$.

$y_0 = 2 \times 10^3 = (1.5)^0 (2 \times 10^3)$

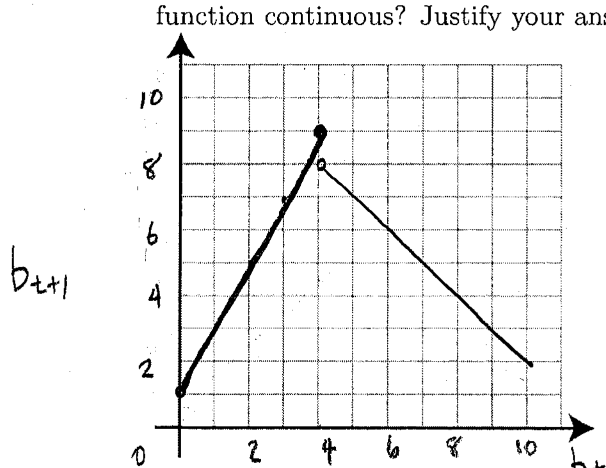
$y_1 = 3000 = 1.5^1 (2 \times 10^3)$

$y_2 = 4500 = 1.5^2 (2 \times 10^3)$ $\boxed{y(t) = 1.5^t (2 \times 10^3)}$

(b) The population b_t of a bacteria colony obeys the discrete-time dynamical system

$$b_{t+1} = \begin{cases} 2b_t + 1, & \text{if } b_t \leq 4 \\ -b_t + 12, & \text{if } b_t > 4 \end{cases}$$

i. Accurately graph the updating function for $0 \leq b_t \leq 10$, labeling your axes. Is the function continuous? Justify your answer using the formal definition of continuity.

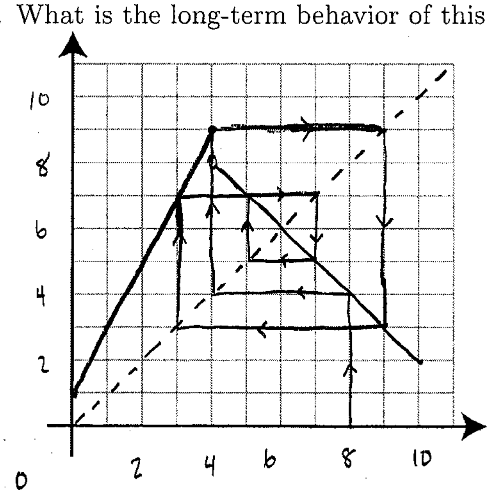


No, it is not continuous because $\lim_{b_t \to 4^-} = 9$

and $\lim_{b_t \to 4^+} = 8$

so $\lim_{b_t \to 4}$ DNE.

ii. Resketch your graph. Also plot the diagonal $b_{t+1} = b_t$. Cobweb for at least seven steps from $b_0 = 8$; clearly label all points (b_t, b_{t+1}) for the first three steps of cobwebbing. What is the long-term behavior of this solution?



The long-term behavior is a 2-cycle.

5. (8 points) Find the following limits. If you use L'Hopital's Rule, justify why it can be applied each time that you use it. If you use knowledge of leading behaviors, justify your work.

(a) $\lim\limits_{x\to\infty} \dfrac{\ln(x^3)}{\ln(x+5)+1}$ form $\frac{\infty}{\infty}$. L'Hôpital's

$$\lim_{x\to\infty} \frac{\frac{1}{x^3}(3x^2)}{\frac{1}{x+5}} = \lim_{x\to\infty} \frac{3(x+5)}{x} \text{ form } \frac{\infty}{\infty} \quad \lim_{x\to\infty} \frac{3}{1} = \boxed{3}$$

(b) $\lim\limits_{x\to 5} \dfrac{x^2-3x-10}{x-5} = \lim\limits_{x\to 5} \dfrac{(x-5)(x+2)}{x-5} = \lim\limits_{x\to 5} x+2 = \boxed{7}$

6. (10 points) Consider the function $f(x) = \frac{2x^3}{1+x^3}$.

(a) Find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$, and $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$.

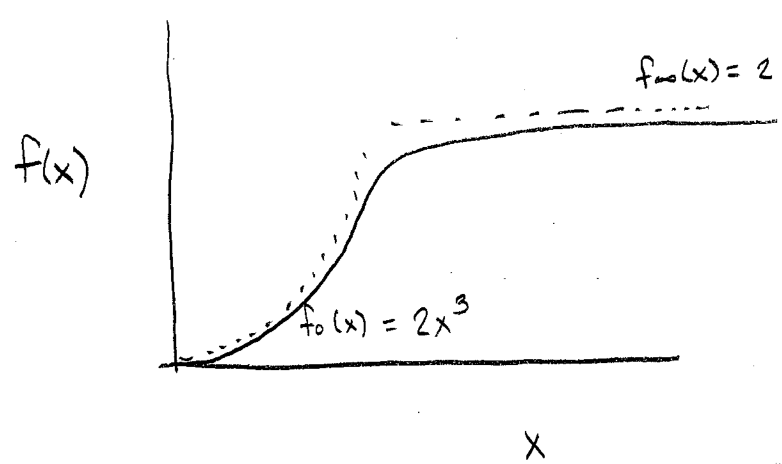
$\text{top}_\infty(x) = 2x^3$
$\text{bot}_\infty(x) = x^3$

$f_\infty(x) = \dfrac{2x^3}{x^3} = 2$

$\text{top}_0(x) = 2x^3$
$\text{bot}_0(x) = 1$

$f_0(x) = \dfrac{2x^3}{1} = 2x^3$

(b) Use the method of matched leading behaviors to sketch a graph of $f(x)$ on the interval $x \geq 0$. Label your axes and indicate where you have graphed $f_\infty(x)$, $f_0(x)$, and $f(x)$.



7. (13 points) After an embarrassing and soul-crushing loss to the Dolphins, Tom Brady develops a severe case of depression. Tom is prescribed a drug to treat his illness. Suppose the probability of Tom's recovering depends on how much of the drug is administered, modeled by

$$P(x) = \frac{2\sqrt{x}}{2+3x}, \qquad 0 \le x \le 10,$$

where x is measured in milligrams (mg), and $P(x)$ is the probability that Tom recovers.

(a) How much of the drug should Tom take in order to maximize the probability that he recovers? Be sure to use Calculus to justify your answer (in particular, justify that your answer gives a *global* maximum, not just a *local* maximum). Don't forget to include units.

$$P'(x) = \frac{(2+3x)\left(2\left(\frac{1}{2}x^{-1/2}\right)\right) - 2\sqrt{x}(3)}{(2+3x)^2}$$

$$= \frac{(2+3x)(x^{-1/2}) - 6\sqrt{x}}{(2+3x)^2}$$

$$= \frac{\frac{2}{\sqrt{x}} - 3\sqrt{x}}{(2+3x)^2} \overset{\text{set}}{=} 0$$

$$\frac{2}{\sqrt{x}} = 3\sqrt{x}$$

$$2 = 3x$$

$$\frac{2}{3} = x$$

mg

First Derivative Test: $\quad + \;\big|_{\frac{2}{3}}\; -$ $\quad$ local max

To show it is a global max:

$P(0) = 0 \qquad P(\frac{2}{3}) \approx .4082 \qquad P(10) \approx .1976$

(b) Rounded to 2 decimal places, what is the probability that he recovers if he takes the amount of the drug found in part (a)?

$$P(2/3) = \frac{2\sqrt{2/3}}{2+3(2/3)} \approx \boxed{.41}$$

8. (14 points) Suppose that the population p_t of bighorn sheep obeys the discrete-time dynamical system

$$p_{t+1} = p_t(k - p_t),$$

where $k > 0$ is a positive parameter.

(a) Find all equilibria. For what values of k is there more than one equilibrium that makes biological sense?

$$p^* = p^*(k-p^*)$$
$$0 = p^*(k-p^*)-p^*$$
$$0 = p^*(k-p^*-1)$$
$$\boxed{0=p^*} \text{ and } \boxed{p^* = k-1}$$

If $k \neq 1$, there will be two equilibria.

(b) For each equilibrium, determine the values of k for which the equilibrium is stable.

$$f(p) = p(k-p) = kp-p^2$$
$$f'(p) = k-2p$$

$p^* = 0$: $f'(0) = k$

$\boxed{-1 < k < 1}$ indicates stability

$p^* = k-1$: $f'(k-1) = k-2(k-1) = k-2k+2 = 2-k$

$-1 < 2-k < 1$ indicates stability

$-3 < -k < -1$

$\boxed{3 > k > 1}$