1. (16 pts) Find the following derivatives and limits. Show all your work. For derivatives, you don't need to simplify. Use parentheses to indicate multiplication where appropriate. For limits, if you use L'Hospital's rule, justify why it can be applied at each time you use it. If you use knowledge of leading behaviors, justify your work.

a.) $f(x) = \ln\left(\frac{x}{x+1}\right)$, $f'(x) = \dfrac{1}{\frac{x}{x+1}}\left(\dfrac{x+1-x}{(x+1)^2}\right)$

b.) $g(t) = e^{\sin(t^2)}$, $g'(t) = \left(e^{\sin(t^2)}\right)\left(\cos(t^2)\right)(2t)$

c.) $f(x) = 6x^7 + x^{-1} + 10^2$, $f'(x) = 42x^6 - x^{-2}$

d.) $\lim_{x\to 1} \frac{2x-2}{\ln(x)}$ Form $\frac{0}{0}$

$$\overset{LR}{=} \lim_{x\to 1} \frac{2}{\frac{1}{x}} = 2$$

e.) $\lim_{t\to\infty} \frac{e^t}{2t+1}$ Form $\frac{\infty}{\infty}$

Using leading behavior: e^t grows faster than any power of t, so the limit is ∞.

Using L'Hospital's rule:

$$\lim_{t\to\infty} \frac{e^t}{2t+1} \overset{LR}{=} \lim_{t\to\infty} \frac{e^t}{2} = \infty$$

2. (12 pts) An individual with a certain disease is given an amount $x > 1$ of a drug where x is measured in ml. Her probability of being cured is given by

$$P(x) = \frac{-x^2}{3x-3} + 2, \quad x > 1$$

Find the value of x on $(1, \infty)$ that maximizes the chances of a cure. Justify your answer using calculus.

$$P'(x) = -\frac{2x(3x-3) - x^2 \cdot 3}{(3x-3)^2} = -\frac{6x^2 - 6x - 3x^2}{(3x-3)^2}$$

$$= -\frac{3x^2 - 6x}{(3x-3)^2} = \frac{-3x(x-2)}{(3x-3)^2} = 0$$

$x = 0,\ x = 2$ are the critical pts

We also have a vertical asymptote at $x = 1$.

By the first derivative test,

$P'(x)$: between 1 and 2: $+ \ +$; after 2: $- \ -$

$$P'\left(\tfrac{1}{2}\right) = -\frac{\frac{3}{2}\left(\frac{1}{2} - 2\right)}{\left(\frac{3}{2} - 3\right)^2} > 0$$

$$P'\left(\tfrac{3}{2}\right) = -\frac{\frac{9}{2}\left(\frac{3}{2} - 2\right)}{\left(\frac{9}{2} - 3\right)^2} > 0$$

$$P'(3) = -\frac{9(1)}{(9-3)^2} < 0$$

$$P(2) = -\frac{4}{12-3} + 2 = -\frac{4}{9} + \frac{18}{9} = \frac{14}{9}$$

So $(2, 14/9)$ is a local max

Since P is decr. for $x > 2$

$x = 2$ ml maximizes the chance of a cure.

3. (14 pts) Consider the function $f(x) = x^3 - 3x^2 + 3x - 9$

a). Give the equation of the tangent line through f(x) at $x = 0$.

$f'(x) = 3x^2 - 6x + 3$

$f'(0) = 3$

$f(0) = -9$

$y = 3x + b$

$-9 = 3(0) + b$

$-9 = b$

$\boxed{y = 3x - 9}$

b). Use Newton's Method to determine the find the solution to $x^3 - 3x^2 + 3x - 9 = 0$ with the initial guess $x_0 = 0$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x_0 = 0$

$x_1 = 0 - \frac{-9}{3} = 3$

$x_2 = 3 - \frac{0}{3} = \boxed{3}$

4. (12 points) Consider the modified version of the logistics equation

$$x_{t+1} = rx_t(1 - x_t^2). \quad , \quad r > 0$$

a) Find all equilibria of the updating function.

$$x^* = rx^*(1-x^{*2})$$

$$0 = r^*(1-x^{*2})-x^*$$

$$0 = x^*(r - rx^{*2} - 1)$$

$$\boxed{x^* = 0} \quad \text{and} \quad rx^{*2} = r-1$$

$$\boxed{x^* = \sqrt{\frac{r-1}{r}}}$$

b) For each equilibrium, determine the values of r for which the equilibrium is stable.

$$f(x) = rx(1-x^2) = rx - rx^3$$

$$f'(x) = r - 3rx^2$$

$$x^* = 0 : \quad f'(0) = r \quad \boxed{-1 < r < 1}$$ indicates stability

$$x^* = \sqrt{\frac{r-1}{r}} : \quad f'\left(\sqrt{\frac{r-1}{r}}\right) = r - 3r\left(\frac{r-1}{r}\right)$$

$$= r - 3r + 3$$

$$= -2r + 3$$

$$-1 < -2r + 3 < 1$$

$$\boxed{2 > r > 1}$$ indicates stability

5. (15 pts) Suppose a coral reef off the coast of Hawaii grows at a rate of $2e^{0.1t} + 0.2t$ centimeters per year. Suppose the year $t = 0$ represents the year 2000, and the reef has height 20 cm in the year 2000.

a) Write a pure-time differential equation with an initial condition to describe the growth $g(t)$ of the coral.

$$\frac{dg}{dt} = 2e^{0.1t} + 0.2t$$

$$g(0) = 20$$

b) How tall will the coral be in the year 2010?

$$\int (2e^{0.1t} + 0.2t)\,dt = \frac{2e^{0.1t}}{0.1} + \frac{0.2}{2}t^2 + c = g(t)$$

$$g(0) = 20e^0 + .1(0) + c = 20$$

$$c = 0$$

$$g(t) = 20e^{0.1t} + .1t^2$$

$$g(10) = 20e^1 + .1(100) \approx \boxed{64.3656 \text{ cm}}$$

c) Suppose the coral will be damaged by snorkelers if it reaches a height of 1000cm. Will this height ever be reached? Use a limit to find the answer. Show all of your work to receive credit.

$$\lim_{t \to \infty} 20e^{0.1t} + .1t^2 = \infty$$

Yes, this height is attainable since the coral will grow without bound.

6. (16 pts) Evaluate the following definite and indefinite integrals. If necessary, use u-substitution.

a) $\int_0^{\pi} x^3 - 5cos(x)\,dx$

$$\frac{x^4}{4} - 5\sin(x)\Big|_0^{\pi} = \boxed{\frac{\pi^4}{4}}$$

b) $\int \frac{8}{t^{\frac{1}{5}}} + e^{3t}\,dt$

$$= \int (8t^{-1/5} + e^{3t})\,dt = \frac{8}{(4/5)} t^{4/5} + \frac{1}{3}e^{3t} + c$$

$$= \boxed{10t^{4/5} + \frac{1}{3}e^{3t} + C}$$

c) $\int x^2 \sin(x^3)\,dx$

$w = x^3$

$\frac{dw}{dx} = 3x^2$

$\frac{dw}{3x^2} = dx$

$$\int x^2 \sin w \left(\frac{dw}{3x^2}\right)$$

$$= \frac{1}{3}\int \sin w\,dw$$

$$= \boxed{-\frac{1}{3}\cos(x^3) + C}$$

d) $\int_2^3 2t(1+t^2)^6\,dt$

$w = 1 + t^2$

$\frac{dw}{dt} = 2t$

$\frac{dw}{2t} = dt$

$$\int 2t(w)^6 \left(\frac{dw}{2t}\right)$$

$$= \int w^6\,dw$$

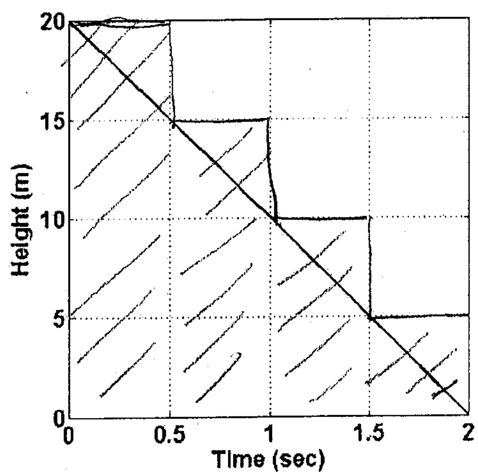
$$= \frac{1}{7}(1+t^2)^7\Big|_2^3$$

$$= \boxed{\frac{10^7}{7} - \frac{5^7}{7}}$$

7. (15 pts) Suppose a stone is thrown upward with initial speed $20m/sec$. Suppose the gravitational acceleration is $10m/sec^2$. Then the velocity of the stone satisfies the differential equation:

$$\frac{dy}{dt} = -10t + 20$$

a) (5 pts) Estimate the total distance the stone traveled between $t = 0$ sec and $t = 2$ sec by using left-hand Riemann sums with $n = 4$. Draw rectangles to label the subintervals you used in the Riemann sums.



Area under the curve approximated by left-hand Riemann sums is

$0.5\,(f(0) + f(.5) + f(1) + f(1.5))$

$= .5\,(20 + 15 + 10 + 5)$

$= 25$

b) (4 pts) Use definite integral to evaluate the total distance the stone traveled between $t = 0$ sec and $t = 2$ sec.

$$\int_0^2 (-10t + 20)\,dt = -10\frac{t^2}{2} + 20t \Big|_0^2$$

$$= -5(4) + (20)(2) = -20 + 40 = 20$$

c) (6 pts) Evaluate the indefinite integral using integration by parts.

$$\int 2xe^x dx$$

$u = 2x \quad dv = e^x$

$du = 2dx \quad v = e^x$

$$2xe^x - \int 2e^x\,dx = 2xe^x - 2e^x + C$$