Math 155 Exam 2. Spring 2011.

1. (15 points) Compute the derivatives of the following functions. You do NOT need to simplify your answer. Use parentheses to indicate multiplication where appropriate, and make sure your notation is correct.

(a) $f(x) = x^2 \tan(x+4)$

$$f'(x) = 2x\tan(x+4) + x^2\sec^2(x+4)\underbrace{\frac{d}{dx}(x+4)}_{=1}$$

$$= \boxed{2x\tan(x+4) + x^2\sec^2(x+4)}$$

(b) $g(x) = \dfrac{\ln(x)}{x^2+1}$

$$g'(x) = \frac{\left(\frac{d}{dx}\ln(x)\right)(x^2+1) - \ln(x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \boxed{\frac{\frac{1}{x}(x^2+1) - 2x\ln(x)}{(x^2+1)^2}}$$

(c) $h(x) = \frac{e^x+\pi}{x^3+2}$

$$h'(x) = \frac{\left(\frac{d}{dx}(e^x+\pi)\right)(x^3+2) - (e^x+\pi)\frac{d}{dx}(x^3+2)}{(x^3+2)^2}$$

$$= \boxed{\frac{e^x(x^3+2) - (e^x+\pi)(3x^2)}{(x^3+2)^2}}$$

(d) $f(t) = (3t^6 + kt + \pi)^7$, where k is a constant.

$$f'(t) = \boxed{7(3t^6 + kt + \pi)^6(18t^5 + k)}$$

(e) $p(x) = e^{\cos(4x)+2}$

$$p'(x) = e^{\cos(4x)+2}\frac{d}{dx}(\cos(4x)+2)$$

$$= \underline{e^{\cos(4x)+2}(-\sin(4x))(4)} = -4\sin(4x)e^{\cos(4x)+2}$$

2. (15 points) Consider a population of coatimundis governed by the discrete-time dynamical system

$$c_{t+1} = rc_t(1 - \frac{1}{2}c_t),\ r > 0.$$

(a) Find all equilibria. For what values of r is there more than one equilibrium that makes biological sense?

$c^* = rc^*(1 - \frac{1}{2}c^*) = rc^* - \frac{r}{2}(c^*)^2$;

$0 = (r-1)c^* - \frac{r}{2}(c^*)^2 = c^*(r - 1 - \frac{r}{2}c^*)$.

Either $\boxed{c^* = 0}$, or $r - 1 - \frac{r}{2}c^* = 0$;

$$\boxed{c^* = \frac{2(r-1)}{r} = 2 - \frac{2}{r}}$$

$c^* = 2 - \frac{2}{r} > 0$ if $2 > \frac{2}{r}$; $r > 1$;

$\boxed{r>1 \text{ is the condition for there to be more than one biologically realistic equilibrium}}$

(b) For $r = 4$, determine the stability of each of the equilibria using the Stability Test/Criterion.

$f(c) = rc(1 - \frac{1}{2}c) = rc - \frac{r}{2}c^2$

$f'(c) = r - rc$

$= (\text{for } r = 4)\ 4 - 4c$

$c^* = 0$: $|f'(0)| = |4 - 4(0)| = |4| > 1$. So, $\underline{c^* = 0 \text{ is unstable}}$

$c^* = 2 - \frac{2}{r} = 2 - \frac{2}{4} = 3/2$: $|f'(3/2)| = |4 - 4(3/2)| = |-2| > 1$, so

$\underline{c^* = 3/2 \text{ is unstable}}$

(c) Determine the values of r for which the nonzero equilibrium is stable.

The condition for $c^* = 2 - 2/r$ to be stable is that

$|f'(2 - 2/r)| < 1$.

$f'(2 - 2/r) = r - r(2 - 2/r) = r - 2r + 2 = 2 - r$, so the condition is

$|2 - r| < 1$; $-1 < 2 - r < 1$;

$-3 < -r < -1$;

$$\boxed{1 < r < 3}$$

3. (10 points) A diver jumps from a diving board. His height (in feet) above the water at time t (in seconds) is given by $h(t) = -16t^2 + 8t + 17$, and he jumps at time $t = 0$.

(a) Find the velocity $v(t)$ and the acceleration $a(t)$.

$v(t) = h'(t) = -32t + 8$

$a(t) = v'(t) = -32$

(b) At what time does the diver reach his maximum height above the water? What is this height?

$h'(t) = -32t + 8 \overset{set}{=} 0$

$t = 8/32 = \underline{1/4}$

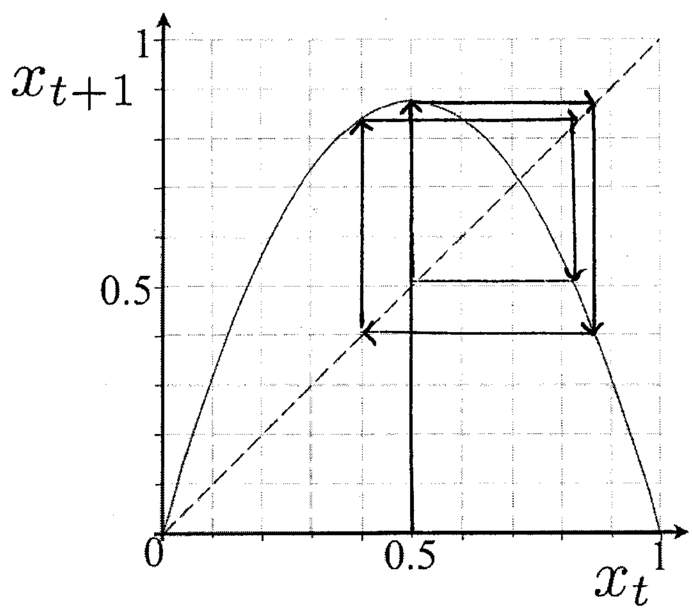
$h''(1/4) = -32 < 0$, so h has a (local) max at $t = 1/4$.

$h(1/4) = -16(1/4)^2 + 8(1/4) + 17$

$= -1 + 2 + 17 = \underline{18}$

4. (5 points) The updating function for the discrete-time dynamical system $x_{t+1} = 3.52x_t(1-x_t)$ is graphed below, along with the diagonal. Cobweb ~~for at least 10 steps~~ starting from $x_0 = 0.5$. What is the long-term behavior of this system?

long-term behavior: 4-cycle



5. (15 points) Consider the function $f(x) = \frac{x^2-3}{e^x}$.

(a) Show that $f'(x) = \frac{-x^2+2x+3}{e^x}$.

$$f'(x) = \frac{(2x)\cancel{e^x} - (x^2-3)\cancel{e^x}}{(e^x)^{\cancel{2}}} = \frac{2x - x^2 + 3}{e^x} = \frac{-x^2+2x+3}{e^x} \quad \checkmark$$

(b) Find all critical points of $f(x)$. Find the intervals in which $f(x)$ is increasing and the intervals in which $f(x)$ is decreasing.

$f'(x) = 0$ if $-x^2+2x+3 = 0$; $x^2-2x-3=0$; $(x+1)(x-3)=0$;

$\boxed{x=-1 \text{ or } x=3}$

$- - - - - - - \;\; -1 \;\; + + + + + + + + + + \;\; 3 \;\; - - - - - -$

$f'(-2) = -5 < 0$ $\quad f'(0) = 3 > 0$ $\quad f'(4) = -5 < 0$

decreasing: $(-\infty, -1)$ and $(3, \infty)$
increasing: $(-1, 3)$

(c) Given that $f''(x) = \frac{x^2-4x-1}{e^x}$ (you do not need to show this), find the intervals in which $f(x)$ is concave up or concave down. Answers without calculus justification will not receive full credit.

$f''(x) = 0$ if $x^2-4x-1=0$;

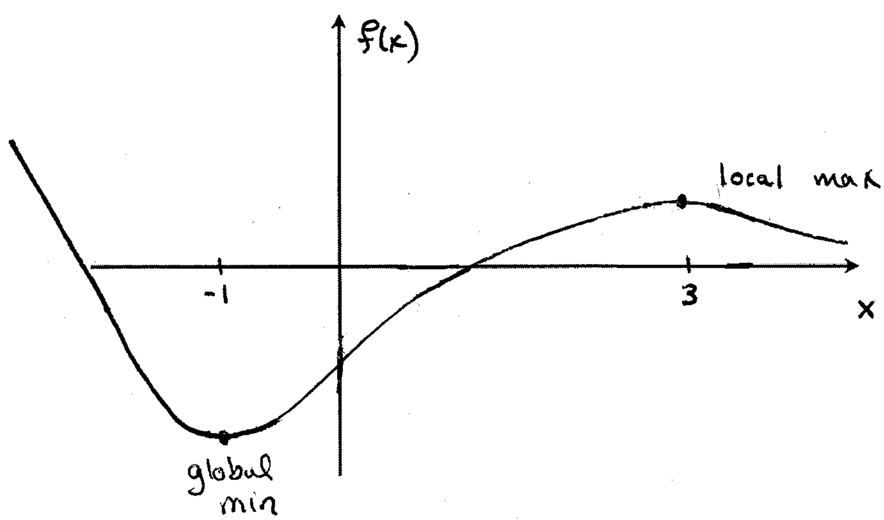
$$x = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

$2-\sqrt{5}$ $\quad 2+\sqrt{5}$

$f''(-10) > 0$ $\quad f''(0) < 0$ $\quad f''(10) > 0$ $\Rightarrow$

concave up on $(-\infty, 2-\sqrt{5}), (2+\sqrt{5}, \infty)$
concave down on $(2-\sqrt{5}, 2+\sqrt{5})$

(d) Use the information found above to sketch a graph of the function $f(x)$. Indicate where any local maxima, local minima, global maxima, or global minima occur.



6. (14 points) Consider the discrete-time dynamical system

$$N_{t+1} = 2N_t(1 - N_t) - hN_t$$

describing a population of fishes being harvested at rate h.

(a) Find the nonzero equilibrium population N^* as a function of h. What is the largest value of h consistent with a ~~positive~~ nonnegative equilibrium?

$N^* = 2N^*(1-N^*) - hN^*$.

We are looking for a nonzero equilibrium, so we can divide by N^*; $1 = 2(1-N^*) - h$; $\frac{1+h}{2} = 1 - N^*$; $N^* = 1 - \frac{1}{2} - \frac{h}{2} = \frac{1}{2}(1-h)$

$\boxed{N^* = \frac{1}{2}(1-h)}$

- $0 \leq N^* = \frac{1}{2}(1-h)$ for $\boxed{h \leq 1}$

$h=1$ is the largest value of h consistent with a nonnegative equilibrium.

(b) The equilibrium harvest is given by $P(h) = hN^*$, where N^* is the equilibrium you found in part (a). Find the value of h that maximizes $P(h)$. Use the first or second derivative test to justify that this value of h gives a local maximum.

$P(h) = hN^* = \frac{1}{2}h(1-h) = \frac{1}{2}h - \frac{1}{2}h^2$

$P'(h) = \frac{1}{2} - h \stackrel{\text{set}}{=} 0$; $\boxed{h = 1/2}$

Since $P''(h) = -1$, $P''(1/2) = -1 < 0$.

The second derivative test implies that $P(1/2)$ is indeed a local max.

7. (14 points) For the following functions, $g(x)$, find $g_\infty(x)$, the leading behavior of $g(x)$ as $x \to \infty$, and $g_0(x)$, the leading behavior of $g(x)$ as $x \to 0$.

$x \to \infty$: $\to 0$, $\to 0$, $\to \infty$, $\to \infty$

(a) $g(x) = 2000x^{-2} + 4e^{-2x} + 100x^3 + 0.4x^7$

$x \to 0$: $\to \infty$, $\to 4$, $\to 0$, $\to 0$

$$\boxed{\begin{aligned} g_\infty(x) &= 0.4x^7 \\ g_0(x) &= 2000x^{-2} \end{aligned}}$$

(b) $g(x) = \dfrac{e^{2x} + x^{-2}}{\ln(x) + 6}$

$top(x) = e^{2x} + x^{-2}$ $\quad top_\infty(x) = e^{2x}$

$top_0(x) = x^{-2}$

$bot(x) = \ln(x) + 6$ $\quad bot_\infty(x) = \ln(x)$

$bot_0(x) = \ln(x)$

$$\boxed{\begin{aligned} g_\infty(x) &= \frac{e^{2x}}{\ln(x)} \\ g_0(x) &= \frac{x^{-2}}{\ln(x)} \end{aligned}}$$

(c) $g(x) = \dfrac{3x}{2 + x^2}$

$top(x) = 3x$ $\quad top_\infty(x) = 3x$

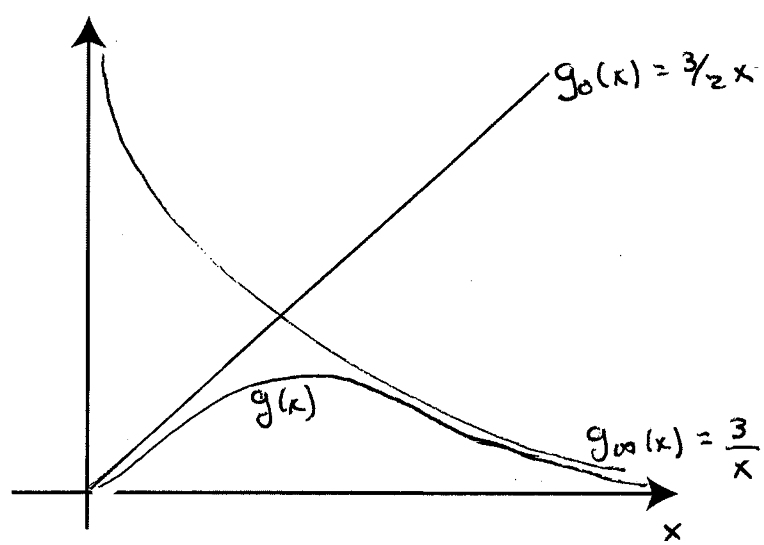
$top_0(x) = 3x$

$bot(x) = 2 + x^2$ $\quad bot_\infty(x) = x^2$

$bot_0(x) = 2$

$$\boxed{\begin{aligned} g_\infty(x) &= \frac{3x}{x^2} = \frac{3}{x} \\ g_0(x) &= \frac{3x}{2} \end{aligned}}$$

(d) For the function in part (c), use the method of matched leading behaviors to sketch the graph of $g(x)$ for $x \geq 0$. Graph and indicate where you have graphed $g_\infty(x)$, $g_0(x)$, and $g(x)$.



8. (12 points) Evaluate the following limits. Show all of your work. If you use leading behavior, justify your answer by explaining all of your steps. If you use L'Hopital's Rule, justify why it can be applied each time you use it.

(a) $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{4x} \overset{\text{L'Hôp}}{=} \lim\limits_{x\to 0} \dfrac{-\sin(x)}{4} = \dfrac{0}{4} = \boxed{0}$

form $\frac{0}{0}$

(b) $\lim\limits_{x\to\infty} \dfrac{\ln(x^2+1)}{x^3+4} \overset{\text{L'Hôp}}{=} \lim\limits_{x\to\infty} \dfrac{\frac{1}{x^2+1}(2\cancel{x})}{3x^{\cancel{2}}} = \lim\limits_{x\to\infty} \dfrac{2}{3x(x^2+1)} = \boxed{0}$

form $\frac{0}{0}$

(c) $\lim\limits_{x\to\infty} \dfrac{\overset{\to\infty}{e^x} + \overset{\to\infty}{x^{100}} + \overset{\to\infty \text{ fastest}}{6e^{5x}}}{\underset{\text{fastest}}{\underset{\to\infty}{2e^{5x}}} + \underset{\to\infty}{x}} = \lim\limits_{x\to\infty} \dfrac{6e^{\cancel{5x}}}{2e^{\cancel{5x}}} = \boxed{3}$

↑ leading behavior