SP09 Exam 2

1. (16 pts) Evaluate the first derivative of the following functions. You do **NOT** need to simplify your answers.

a) $s(x) = x^9 + 4x^3 - 5x^2 + \frac{1}{6}x^{-1/2} - 10^{5/8} + \pi$

$$s'(x) = 9x^8 + 12x^2 - 10x - \frac{1}{12}x^{-3/2}$$

b) $g(x) = e^{9x}\sin(\alpha^2 x^7)$ where α is a constant.

$$g'(x) = e^{9x}\cos(\alpha^2 x^7)(7x^6) + \sin(\alpha^2 x^7)e^{9x}(9)$$

c) $p(x) = \dfrac{\ln(2x^3)}{\tan(3x)}$

$$p'(x) = \frac{\tan(3x)\left(\frac{1}{2x^3}\right)(6x^2) - \ln(2x^3)\sec^2(3x)(3)}{\tan^2(3x)}$$

d) $g(x) = (2x^5 + 17)(3x^{-2} - 9)$

$$g'(x) = (2x^5 + 17)(-6x^{-3}) + (3x^{-2} - 9)(10x^4)$$

2. (12 pts) Consider a population of fish governed by the logistic dynamical system with harvesting rate h where $0 \leq h \leq 1$

$$x_{t+1} = 2x_t(1 - x_t) - hx_t.$$

a) Find all equilibria.

$$x^* = 2x^*(1-x^*) - hx^*$$

$$0 = 2x^*(1-x^*) - hx^* - x^*$$

$$0 = x^*(2 - 2x^* - h - 1)$$

$$\boxed{0 = x^*} \quad \text{and} \quad 1-h = 2x^*$$

$$\boxed{\frac{1-h}{2} = x^*}$$

b) Use the stability criterion to determine all values of h for which the nonzero equilibrium is stable. State or show clearly how you are using the criterion to receive full credit.

$$f(x) = 2x(1-x) - hx = 2x - 2x^2 - hx$$

$$f'(x) = 2 - 4x - h$$

$$f'\left(\frac{1-h}{2}\right) = 2 - 4\left(\frac{1-h}{2}\right) - h = 2 - 2(1-h) - h$$

$$= 2 - 2 + 2h - h = h$$

$-1 < h < 1$ indicates stability.

Since $h \geq 0$, we have $0 \leq h < 1$.

3. (14 pts) Given the updating function

$$x_{t+1} = x_t^2 - 1$$

a) Determine the long-term behavior of the system by finding x_1, x_2, x_3, x_4 and x_5. Begin at $x_0 = 1$. What is the long-term behavior of the system?

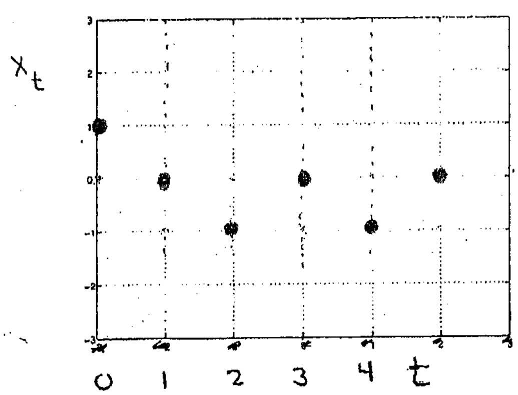
$X_0 = 1$ $\qquad X_4 = -1$

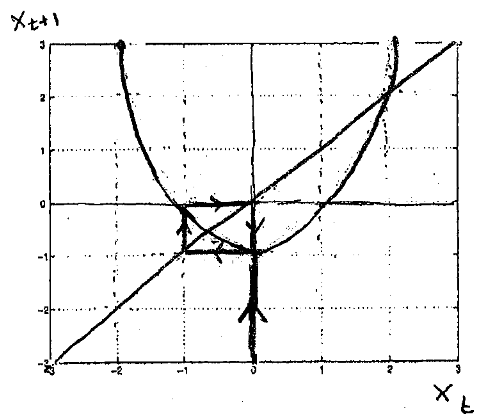
$X_1 = 0$ $\qquad X_5 = 0$

$X_2 = -1$ $\qquad$ The long-term behavior:

$X_3 = 0$ $\qquad$ oscillations between -1 and 0.

b) Draw your iterates $x_0, x_1, \ldots, x_5$on the graph on the left (a graph of x_t versus t). Draw a cobweb diagram on the graph to the right starting this time from $\boxed{x_0 = 0}$. In each case, label your axes.





c) Find any equilibria. Show all of your work. Based on the results of parts (a) and (b), do you expect each of them to be stable or unstable?

$$X^* = X^{*2} - 1$$

$$0 = X^{*2} - X^* - 1$$

$$X^* = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

We expect these equilibria to be unstable from our cobwebbing.

4. (16 pts) Let $f(x) = \frac{1}{3}x^3 - \frac{1}{6}x^6$ on $(-\infty, \infty)$.

a) Find the critical points, showing all your work and using calculus to justify your answer.

$$f'(x) = x^2 - x^5 \overset{\text{set}}{=} 0$$
$$x^2(1-x^3) = 0$$
$$x^2 = 0 \text{ and } 1 = x^3$$
$$\boxed{x=0} \text{ and } \boxed{x=1}$$

b) List the intervals where the function is increasing and decreasing, showing all your work and using calculus to justify your answer.

$f'(x) =$ + (0) + (1) −

$f'(-1) = 1(2) > 0$

$f'(\frac{1}{2}) = \frac{1}{4}(1-\frac{1}{8}) > 0$

$f'(2) = 4(1-8) < 0$

f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$

c) Find $f''(x)$

$$f''(x) = 2x - 5x^4$$

d) List intervals where the function is concave up and concave down, showing all your work and using calculus to justify your answer.

$$f''(x) = 2x - 5x^4 \overset{\text{set}}{=} 0$$
$$x(2-5x^3) = 0$$
$$x = 0 \text{ and } 2 = 5x^3$$
$$\frac{2}{5} = x^3$$
$$\sqrt[3]{2/5} = x \approx .7368$$

$f''(-1) = -1(2+5) < 0$

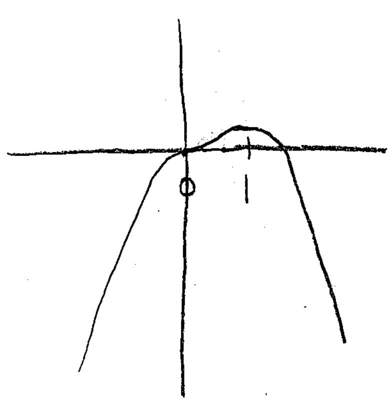
$f''(\frac{1}{2}) = \frac{1}{2}(2-5(\frac{1}{8})) > 0$

$f''(1) = 1(2-5) < 0$

concave up: $(0, .7368)$

concave down: $(-\infty, 0)$ $(.7368, \infty)$

e) Sketch a curve of $f(x)$ using your previous work.



5. (16 pts) Evaluate the following limits. Show all of your work. If you use L'Hôpital's Rule, justify why it can be applied each time you use it. If you use knowledge of leading behavior, justify your work by explaining all of your steps.

a) $\lim_{x\to\infty} \dfrac{\ln x + x}{e^x + x^2}$

form $\frac{\infty}{\infty}$ L'Hôpital's

$$\lim_{x\to\infty} \frac{\frac{1}{x}+1}{e^x+2x} = \lim_{x\to\infty} \frac{x+1}{x(e^x+2x)}$$

form $\frac{\infty}{\infty}$

$$\lim_{x\to\infty} \frac{1}{x(e^x+2)+e^x+2x} = \boxed{0}$$

b) $\lim_{x\to 0} \dfrac{x\cos x}{\tan x}$

form $\frac{0}{0}$ L'Hôpital's

$$\lim_{x\to 0} \frac{x(-\sin x)+\cos x}{\sec^2(x)} = \frac{0+1}{1} = \boxed{1}$$

c) $\lim_{x\to 0} \dfrac{3x^2+x^3}{x^4-x}$

$$= \lim_{x\to 0} \frac{3x+x^2}{x3-1} = \frac{0}{-1} = \boxed{0}$$

d) $\lim_{x\to 1} \dfrac{e^{(x-1)}}{\sqrt{x}\cos(x-1)}$

$$= \frac{e^0}{\sqrt{1}\,(\cos(0))} = \frac{1}{1(1)} = \boxed{1}$$

6. (12 pts) For the following functions $f(x)$, find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$, and $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$

$\to\infty \quad \nearrow 0 \quad \nearrow 0 \quad \nearrow 0 \quad \nearrow 0$

a) $f(x) = e^{-4x} + \frac{10}{x^2} + \frac{3}{x^5} + 12e^{-2x}$

$\to 0 \quad \searrow 1 \quad \searrow \infty \quad \searrow \infty \quad \searrow 12$

$f_\infty(x) = \frac{10}{x^2}$

$f_0(x) = \frac{3}{x^5}$

b) $f(x) = \frac{4x+2}{e^x - x}$

$top_\infty(x) = 4x$

$bot_\infty(x) = e^x$

$top_0(x) = 2$

$bot_0(x) = e^x$

$f_\infty(x) = \frac{4x}{e^x}$

$f_0(x) = \frac{2}{e^x}$

c) $f(x) = \frac{6x}{1+x}$

$top_\infty(x) = 6x$

$bot_\infty(x) = x$

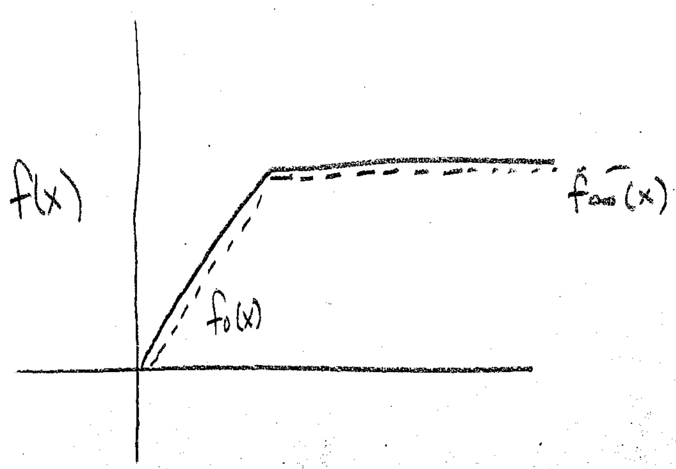
$top_0(x) = 6x$

$bot_0(x) = 1$

$f_\infty(x) = \frac{6x}{x} = 6$

$f_0(x) = \frac{6x}{1} = 6x$

d) For the function in part (c), use the method of matched leading behaviors to sketch a graph of $f(x)$ on the interval $[0, \infty)$.



7. (14 pts) Suppose the size of a bird population in hundreds is governed by $h(x) = \dfrac{3+x^2}{x}$ for x in $[1, 10]$ where x represents the number (in thousands) of a species of tree borer that serves as its food supply.

a) Find the critical points and classify them as local minima or maxima using the *second derivative*.

$$h'(x) = \frac{2x^2 - (3+x^2)}{x^2} = \frac{2x^2 - 3 - x^2}{x^2} = \frac{x^2-3}{x^2}$$

$x^2 - 3 = 0 \Rightarrow x = \pm\sqrt{3}$ are the critical pts

$$h''(x) = \frac{2x(x^2) - (x^2-3)(2x)}{x^4} = \frac{2x^3 - 2x^3 + 6x}{x^4}$$

$$= \frac{6}{x^3}$$

$h(\sqrt{3}) > 0$ So $\sqrt{3}$ is a local min.

b) Find the values of x for which h attains a global minimum and for which h attains a global maximum. Justify your answers.

$$h(1) = \frac{3+1}{1} = 4$$

$$h(\sqrt{3}) = \frac{3+3}{\sqrt{3}} = \frac{2(\sqrt{3})^2}{\sqrt{3}} = 2\sqrt{3} \approx 3.46$$

$$h(10) = \frac{103}{10}$$

So h attains a global min at $x = \sqrt{3}$ and h attains a global max at $x = 10$.

c) What is the meaning of your answers to part b) in terms of the bird population and its food supply? If the food supply is at its max. val of $x = 10$ (10,000 tree borers), the size of the bird pop. will be at a max. If the food supply is at $x = \sqrt{3}$ ($\approx$ 1732 tree borers) the bird pop will sink to the min. level of 346 birds.