Math 155 **Exam 2** **Fall 2009**

NAME: ____________________________________

SECTION: __________________ TIME: __________________

INSTRUCTOR: ____________________________________

Instructions: The exam is closed book and closed notes. You may use an approved calculator, but be sure to show your work on each problem for full credit. Work that is crossed out or erased will not be graded. Turn in any scratch paper that you use during the exam. You will have one hour and 45 minutes to work on the exam.

Problem	Points	Score
1	15	
2	15	
3	14	
4	14	
5	14	
6	16	
7	12	
Total	100	

CONFIDENTIALITY PLEDGE

I agree that I will not share any information, either specific or general, about the problems on this examination with any other person until the exams have been returned to us in class.

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(Signature)

1. (15 points) Find the derivatives of the following functions. You do not need to simplify your answer, but do use proper notation.

   (a) $f(x) = (1 + x^3)^4$

   $f'(x) = 4(1+x^3)^3\,(3x^2)$

   (b) $g(x) = x^{-1/2} + x + e^{2x} + \ln(3)$

   $g'(x) = -\frac{1}{2}x^{-3/2} + 1 + 2e^{2x}$

   (c) $h(x) = \ln(x) \cdot \sin(x) - x^4$

   $h'(x) = \ln(x)\cos(x) + \sin(x)\left(\frac{1}{x}\right) - 4x^3$

   (d) $m(x) = \frac{x}{3x+2}$

   $$m'(x) = \frac{(3x+2)(1) - x(3)}{(3x+2)^2}$$

   (e) $k(x) = e^{\sec(x)}$ (Express your final answer without $\sin(x)$ or $\cos(x)$.)

   $k'(x) = e^{\sec(x)}\sec(x)\tan(x)$

3. (14 points) Consider the following logistic dynamical system:

$$x_{t+1} = 2.5x_t(1 - x_t)$$

(a) Find the equilibria algebraically.

$$x^* = 2.5x^*(1-x^*)$$
$$0 = 2.5x^*(1-x^*) - x^*$$
$$0 = x^*\ (2.5(1-x^*) - 1)$$
$$\boxed{x^* = 0} \quad \text{and} \quad 2.5 - 2.5x^* - 1 = 0$$
$$\boxed{x^* = .6}$$

(b) Use the Stability Test to determine the stability of each equilibrium point.

$$f(x) = 2.5x(1-x) = 2.5x - 2.5x^2$$
$$f'(x) = 2.5 - 5x$$
$$x^* = 0: \quad f'(0) = |2.5| > 1 \quad \text{unstable}$$
$$x^* = .6: \quad f'(.6) = 2.5 - 5(.6) = |-.5| < 1 \quad \text{stable}$$

(c) Use the initial condition $x_0 = 0.25$ to calculate x_1, x_2, x_3, x_4, and x_5. Then cobweb for five time-steps from the initial condition $x_0 = 0.25$.

$X_0 = 0.25$

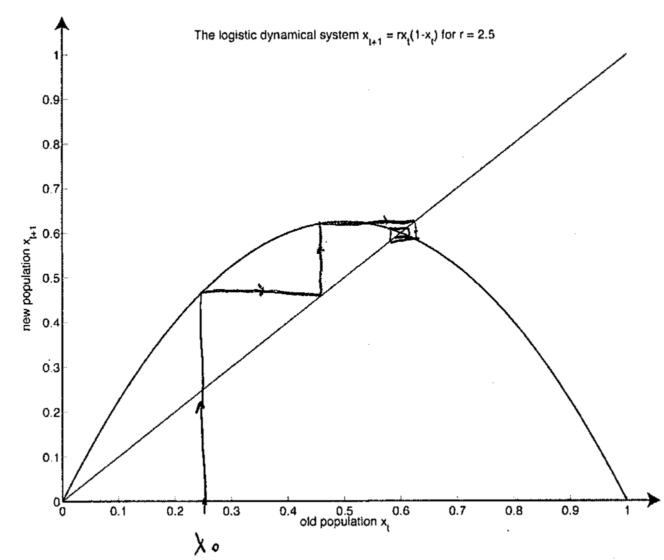
$X_1 = 0.46875$

$X_2 \approx .6226$

$X_3 \approx .5874$

$X_4 \approx .6059$

$X_5 = .5970$



(d) What is the long-term behavior of the system?

goes to the equilibrium $x^* = 0.6$

2. (15 points) Consider the function $f(x) = x^3 - 10.5x^2 + 30x + 15$ on the interval $[0, 8]$.

(a) Calculate $f'(x)$, and use this to find all the critical points of $f(x)$.

$$f'(x) = 3x^2 - 21x + 30 \overset{\text{set}}{=} 0$$

$$3(x^2 - 7x + 10) = 0$$

$$(x-5)(x-2) = 0$$

$$x = 5 \text{ and } x = 2$$

(b) Calculate $f''(x)$, and use this to find any inflection points of $f(x)$.

$$f''(x) = 6x - 21 \overset{\text{set}}{=} 0$$

$$6x = 21$$

$$x = \frac{21}{6} = 3.5$$

Second Derivative Test

$-$ | $+$

3.5

change in concavity

(c) Classify each critical point as a local maximum, local minimum or neither. Justify your answer using the first or second derivative test. Write your answer in coordinate form $((x, f(x)))$.

First Derivative Test: $+$ | $-$ | $+$ (at 2 and 5)

(2, 41) local max

(5, 27.5) local min

(d) Use the information found above to carefully sketch a graph of $f(x)$ on the interval $[0, 8]$. Indicate where any local maxima, local minima, global maxima, global minima, or inflection points occur.

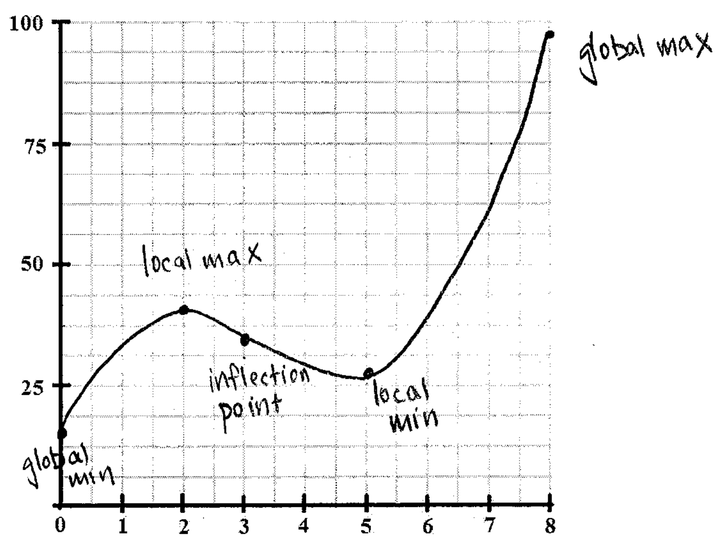
$f(0) = 15$

$f(2) = 41$

$f(5) = 27.5$

$f(8) = 95$

$f(3.5) = 34.25$



4. (14 points) Suppose that the population x_t of a colony of bacteria satisfies the discrete-time dynamical system

$$x_{t+1} = \frac{x_t}{r + x_t^2},$$

where r is a parameter, and $0 < r < 1$.

(a) Verify algebraically that the positive equilibrium is $x^* = \sqrt{1-r}$.

$$\sqrt{1-r} = \frac{\sqrt{1-r}}{r + (\sqrt{1-r})^2} = \frac{\sqrt{1-r}}{r+1-r} = \sqrt{1-r} \checkmark$$

(b) Show that the derivative of the updating function is $\frac{r-x^2}{(r+x^2)^2}$.

$$f(x) = \frac{x}{r+x^2}$$

$$f'(x) = \frac{(r+x^2)(1) - x(2x)}{(r+x^2)^2} = \frac{r+x^2-2x^2}{(r+x^2)^2} = \frac{r-x^2}{(r+x^2)^2} \checkmark$$

(c) Find a condition on r so that $x^* = \sqrt{1-r}$ is a stable equilibrium.

$$f'(\sqrt{1-r}) = \frac{r-(\sqrt{1-r})^2}{(r+(\sqrt{1-r})^2)^2} = \frac{r-1+r}{(r+1-r)^2} = 2r-1$$

$|2r-1| < 1$ indicates stability

$$-1 < 2r-1 < 1$$

$$0 < 2r < 2$$

$$\boxed{0 < r < 1}$$

5. (14 points) In the upcoming Patriots-Dolphins game this weekend, Tom Brady throws an interception to Vontae Davis who then returns it for a touchdown, sealing the victory for Miami. Suppose the height $h(t)$ (in yards) of the football above the ground at time t (in seconds) during the pass is given by

$$h(t) = -5t^2 + 18t + 2, \quad 0 \leq t \leq 3.61.$$

(a) At what time t does $h(t)$ reach its global maximum? How high above the ground is the ball at this time? Be sure to use Calculus to justify your answer, and don't forget to include the units.

$$h'(t) = -10t + 18 \overset{\text{set}}{=} 0$$

$$-10t = -18$$

$$t = 1.8 \text{ seconds}$$

$h(1.8) = 18.2$ yards

First Derivative Test:

$$\underset{+}{\longrightarrow} \; 1.8 \; \underset{-}{\longrightarrow}$$

1.8 is where $h(t)$ attains a max.

(b) At what time t does $h(t)$ reach its global minimum? How high above the ground is the ball at this time? Again, justify your answer and don't forget to include units.

$h(0) = 2$ yards

$h(1.8) = 18.2$ yards

$h(3.61) = 1.8195$ yards

$t = 3.61$ seconds is the global min

(c) Find the velocity $v(t)$ and the acceleration $a(t)$ of the football as functions of time.

$v(t) = h'(t) = -10t + 18$ yards/sec

$a(t) = h''(t) = -10$ yards/sec^2

6. (16 pts) Evaluate the following limits. Show all of your work. If you use L'Hospital's Rule, justify why it can be applied each time you use it. If you use knowledge of leading behaviors, justify your work by explaining all of your steps.

(a) $\lim_{x\to\infty} \frac{5x^2 - x - 9}{10 + x + 6x^2}$ form $\frac{\infty}{\infty}$ L'Hôpital's

$$\lim_{x\to\infty} \frac{10x - 1}{12x + 1} \text{ form } \frac{\infty}{\infty} \quad \lim_{x\to\infty} \frac{10}{12} = \boxed{\frac{5}{6}}$$

(b) $\lim_{x\to 0} \frac{e^{3x} + x^2 - 10}{x^{-5} + e^{-2x}}$

$e^{3x} \nearrow 1$, $x^2 \nearrow 0$, $-10 \nearrow 10$, $x^{-5} \searrow \infty$, $e^{-2x} \searrow 1$

Leading Behavior

$$\text{top}_0(x) = e^{3x} - 10$$
$$\text{bot}_0(x) = x^{-5}$$

$$\lim_{x\to 0} \frac{e^{3x} - 10}{x^{-5}} = \lim_{x\to 0} x^5(e^{3x} - 10) = \boxed{0}$$

(c) $\lim_{x\to 10} \frac{x^2 - 7x - 30}{x^2 - 6x - 27}$

$$= \lim_{x\to 10} \frac{(x-10)\cancel{(x+3)}}{(x-9)\cancel{(x+3)}} = \lim_{x\to 10} \frac{x-10}{x-9} = \boxed{0}$$

(d) $\lim_{x\to\infty} \frac{\ln(x+3)}{\ln(5x^2)}$ form $\frac{\infty}{\infty}$ L'Hôpital's

$$\lim_{x\to\infty} \frac{\frac{1}{x+3}}{\frac{1}{5x^2}(10x)} = \lim_{x\to\infty} \frac{1}{x+3}\left(\frac{x}{2}\right) = \lim_{x\to\infty} \frac{x}{2x+6}$$

$$\text{form } \frac{\infty}{\infty} \quad \lim_{x\to\infty} \frac{1}{2} = \boxed{\frac{1}{2}}$$

7. (12 points) For the following functions $f(x)$, find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$, and $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$.

$\to\infty \quad \to 0 \quad \to 0 \quad \to 0 \quad \to 0$

(a) $f(x) = 11e^{-3x} + 2x^{-2} + 4x^{-5} + .5e^{-x}$

$\to 0 \quad \searrow 11 \quad \searrow \infty \quad \searrow \infty \quad \searrow .5$

$f_\infty(x) = 2x^{-2}$

$f_0(x) = 4x^{-5}$

(b) $f(x) = \frac{9x^3+8x^{-1}}{3x^2+2}$

$top_\infty(x) = 9x^3$

$bot_\infty(x) = 3x^2$

$f_\infty(x) = \frac{9x^3}{3x^2} = 3x$

$top_0(x) = 8x^{-1}$

$bot_0(x) = 2$

$f_0(x) = \frac{8x^{-1}}{2} = \frac{4}{x}$

(c) For the function in part (b), use the method of matched leading behaviors to sketch a graph of $f(x)$ on the interval $x \geq 0$. Label your axes and indicate where you have graphed $f_\infty(x)$, $f_0(x)$, and $f(x)$.