Math 155
Exam 2
Fall 2012

1. (15 points) Compute the derivatives of the following functions. You do NOT need to simplify your answer. Use parentheses to indicate multiplication where appropriate, and make sure your notation is correct.

   (a) $f(t) = \ln(10) - 2t^4 + \sqrt[5]{t^4} + 9t^{-\frac{1}{3}}$ $= \ln(10) - 2t^4 + t^{4/5} + 9t^{-1/3}$

   $\frac{df}{dt} = -8t^3 + \frac{4}{5}t^{-1/5} - 3t^{-4/3}$

   (b) $g(x) = 2^x(x^3 + \pi)$

   $\frac{dg}{dx} = 2^x(3x^2) + \ln(2)2^x(x^3 + \pi)$

   (c) $r(x) = 3e^{x\sin(x)+2}$

   $\frac{dr}{dx} = 3e^{x\sin(x)+2}\frac{d}{dx}(x\sin(x)+2)$

   $= 3e^{x\sin(x)+2}(x\cos(x) + 1\cdot\sin(x))$

   $= 3(x\cos(x) + \sin(x))e^{x\sin(x)+2}$

   (d) $q(u) = \frac{(u^2+1)^{12}}{\cos(u)}$

   $\frac{dq}{du} = \dfrac{\cos(u)\cdot 12(u^2+1)^{11}(2u) - (u^2+1)^{12}\cdot(-\sin(u))}{\cos^2(u)}$

   $= \dfrac{24u(u^2+1)^{11}\cos(u) + (u^2+1)^{12}\sin(u)}{\cos^2(u)}$

   (e) $p(t) = \tan(\ln(t) + k)$, where k is a constant.

   $p'(t) = \sec^2(\ln(t) + k)\cdot\frac{1}{t}$

2. (15 points) Suppose that the population b_t of boreal owls satisfies the discrete-time dynamical system

$$b_{t+1} = \frac{3b_t}{r + 2b_t},$$

where r is a positive parameter.

(a) Find all equilibria of the discrete-time dynamical system. For what range of values of r is there a positive equilibrium?

$$b^* = \frac{3b^*}{r+2b^*}$$

$\boxed{b^* = 0}$, or $\quad 1 = \dfrac{3}{r+2b^*}$;

$r + 2b^* = 3$

$2b^* = 3 - r$

$\boxed{b^* = \dfrac{3-r}{2}}$

$\boxed{\text{There is a positive equilibrium for } r < 3.}$

(b) Show that the derivative of the updating function is $\frac{3r}{(r+2b)^2}$.

$$f(b) = \frac{3b}{r+2b}$$

$$f'(b) = \frac{(r+2b)(3) - 3b(2)}{(r+2b)^2}$$

$$= \frac{3r + 6b - 6b}{(r+2b)^2} = \frac{3r}{(r+2b)^2}.$$

(c) For each equilibrium, use the Stability Criterion/Stability Theorem to find the range of r for which that equilibrium is stable.

$\underline{b^* = 0}$ is stable if $\quad |f'(0)| < 1$;

$$\left|\frac{3r}{[r+2(0)]^2}\right| < 1; \quad -1 < \frac{3r}{r^2} < 1; \quad -1 < \frac{3}{r} < 1;$$

r is a <u>positive</u> parameter, so we consider only $\frac{3}{r} < 1$; $\quad \underline{r > 3}$;

$\boxed{b^* = 0 \text{ is stable if } r > 3}$.

$\underline{b^* = \frac{3-r}{2}}$ is stable if $\quad \left|f'\left(\frac{3-r}{2}\right)\right| < 1$; $\quad \left|\dfrac{3r}{\left[r+2\left(\frac{3-r}{2}\right)\right]^2}\right| < 1$; $\quad \left|\dfrac{3r}{3^2}\right| < 1$;

$$\left|\frac{r}{3}\right| < 1;$$

$$-1 < \frac{r}{3} < 1.$$

r is a <u>positive</u> parameter, so we consider only $\frac{r}{3} < 1$; $\quad \underline{r < 3}$.

$\boxed{b^* = \frac{3-r}{2} \text{ is stable if } r < 3}$

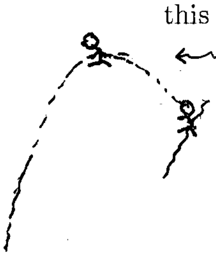
3. (8 points) Jumpie the Monkey jumps from a tree branch down to the ground. Her height (in meters) above the ground at time t (in seconds) is given by $h(t) = -4.9t^2 + 3t + 8$, and she jumps at time $t = 0$.

(a) Find the velocity $v(t)$ and the acceleration $a(t)$ of the monkey at time $t = 1$.

$v(t) = h'(t) = -9.8t + 3$; $\boxed{v(1) = -9.8(1) + 3 = -6.8\ \frac{m}{sec}}$

$a(t) = v'(t) = -9.8$; $\boxed{a(1) = -9.8\ \frac{m}{sec^2}}$

(b) At what time does the monkey reach her maximum height above the ground? What is this height? Use Calculus to justify your answer.



maximum height occurs at a critical point for $h(t)$.

$h'(t) = -9.8t + 3 \overset{set}{=} 0$:

$t = \frac{3}{9.8}$.

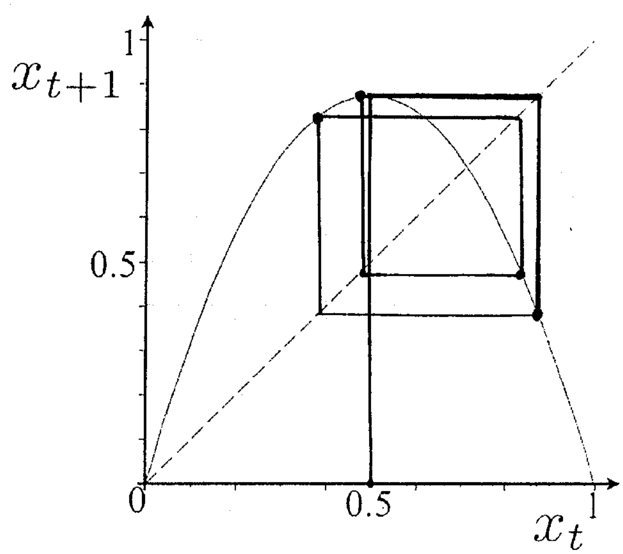
$h\left(\frac{3}{9.8}\right) = -4.9\left(\frac{3}{9.8}\right)^2 + 3\left(\frac{3}{9.8}\right) + 8$ $\boxed{\approx 8.459\ m}$

There is a local max at $t = 3/9.8$ since $h''(t = 3/9.8) = -9.8 < 0$.

This is in fact a global max since $h(t=0) = 8 < 8.459$.

4. (5 points) The updating function for the discrete-time dynamical system $x_{t+1} = 3.52x_t(1-x_t)$ is graphed below, along with the diagonal. Cobweb for at least 8 steps starting from $x_0 = 0.5$. What is the long-term behavior of this system?

The system settles into a 4-cycle as its long-term behavior.



5. (15 points) Consider the function $f(x) = x^3 - 3x^2 - 9x + 2$ on the interval $[-2, 6]$.

(a) Calculate $f'(x)$, and use this to find all the critical points of $f(x)$.

$f'(x) = 3x^2 - 6x - 9$

Critical points: $f'(x) = 3x^2 - 6x - 9 \overset{\text{set}}{=} 0;$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$\boxed{x = 3 \text{ and } x = -1}$

(b) Calculate $f''(x)$, and use this to find regions where $f(x)$ is concave up or concave down.

$f''(x) = 6x - 6$

inflection point(s): $6x - 6 \overset{\text{set}}{=} 0 \rightarrow \underline{x = 1}.$

For $x < 1$, $f''(x) = 6x - 6$ is < 0, so $f(x)$ is concave down for $x < 1$.

For $x > 1$, $f''(x) = 6x - 6$ is > 0, so $f(x)$ is concave up for $x > 1$.

(c) For each critical point, determine if $f(x)$ has a local maximum or a local minimum there. Justify your answer using the first or second derivative test.

<u>2nd derivative test:</u>

$x = -1$: $f''(-1) = 6(-1) - 6 = -12 < 0 \Rightarrow$ $\boxed{f \text{ has a local maximum at } x = -1.}$

$x = 3$: $f''(3) = 6(3) - 6 = 12 > 0 \Rightarrow$ $\boxed{f \text{ has a local minimum at } x = 3.}$

<u>1st derivative test:</u>

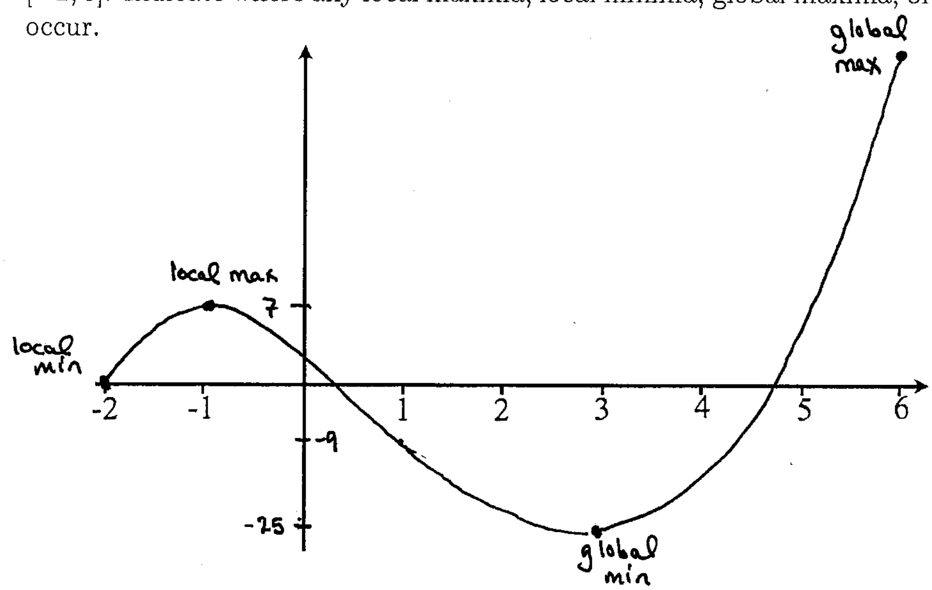
$+ + + + + + + \mid - - - - - - - - - \mid + + + + + + +$

-2 $\quad \boxed{x=-1}$ $\quad 0$ $\quad \boxed{x=3}$ $\quad 4$

$f'(-2) = 15 > 0$ $\quad f'(0) = -9 < 0$ $\quad f'(4) = 15 > 0$

$\boxed{\text{local max at } x = -1}$ $\quad \boxed{\text{local min at } x = 3}$

(d) Use the information found above to sketch a graph of the function $f(x)$ on the interval $[-2, 6]$. Indicate where any local maxima, local minima, global maxima, or global minima occur.



$f(-1) = 7$

$f(3) = -25$: global min

$f(-2) = 0$

$f(6) = 56$: global max

$f(1) = -9$

6. (15 points) Consider the discrete-time dynamical system

$$M_{t+1} = M_t(10 - h - M_t) - hM_t$$

describing a population M_t of mussels in a lake being harvested at rate h. In this model, the activity related to harvesting affects the carrying capacity of the population.

(a) Find the nonzero equilibrium population M^* as a function of h. What is the largest value of h consistent with a nonnegative equilibrium?

$M^* = M^*(10-h-M^*) - hM^*$

For the nonzero M^*, $1 = (10-h-M^*) - h$;

$1+h = 10-h-M^*$

$\boxed{M^* = 9-2h}$

The largest value of h for which $M^* \geq 0$ is h such that

$9-2h = 0$;

$\boxed{h = 9/2}$

(b) The equilibrium harvest is given by $P(h) = hM^*$, where M^* is the equilibrium you found in part (a). Find the value of h that maximizes $P(h)$. Use the first or second derivative test to justify that this value of h gives a local maximum.

$P(h) = h(9-2h) = 9h - 2h^2$.

Critical points of $P(h)$:

$P'(h) = 9 - 4h \overset{\text{set}}{=} 0$;

$4h = 9$; $h = 9/4$

Is there a local max or local min at $h = 9/4$?

2nd derivative test: Check $P''(h=9/4)$: $P''(h) = -4$;

$P''(9/4) = -4 < 0 \Rightarrow$ There is a local max at $h = 9/4$.

In fact, since $P(0) = 0$, (0 is an end point for the domain) there is

$\boxed{\text{a global max at } h = 9/4}$

7. (14 points) For the following functions, $f(x)$, find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$, and $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$.

$x \to \infty$: $\nearrow 0$, $\nearrow \infty$, $\nearrow \infty$, $\nearrow \infty$, $\nearrow 0$

(a) $f(x) = e^{-5x} + 100x^4 + 30x^2 + e^{2x} + x^{-1}$

$x \to 0$: $\nearrow 1$, $\nearrow 0$, $\nearrow 0$, $\nearrow 1$, $\nearrow \infty$

$f_\infty(x) = e^{2x}$

$f_0(x) = x^{-1}$

(b) $f(x) = \dfrac{5x^3 + x^{-2}}{10 + x^{-1} + x^2}$

$top(x) = 5x^3 + x^{-2}$ $\quad top_\infty(x) = 5x^3$

$top_0(x) = x^{-2}$

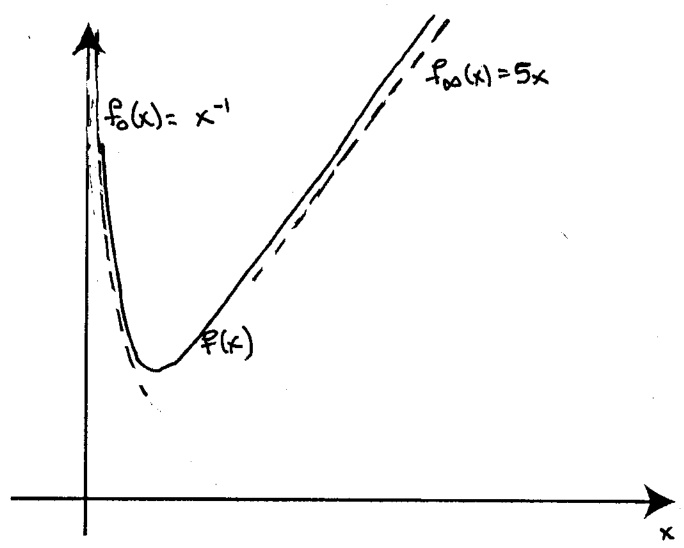
$bot(x) = 10 + x^{-1} + x^2$ $\quad bot_\infty(x) = x^2$

$bot_0(x) = x^{-1}$

$$f_\infty(x) = \frac{top_\infty(x)}{bot_\infty(x)} = \frac{5x^3}{x^2} = 5x$$

$$f_0(x) = \frac{top_0(x)}{bot_0(x)} = \frac{x^{-2}}{x^{-1}} = x^{-1}$$

(c) For the function in part (b), use the method of matched leading behaviors to sketch the graph of $f(x)$ for $x \geq 0$. Graph and indicate where you have graphed $f_\infty(x)$, $f_0(x)$, and $f(x)$.



8. (13 points) Evaluate the following limits. Show all of your work. If you use leading behavior, justify your answer by explaining all of your steps. If you use L'Hopital's Rule, justify why it can be applied each time you use it.

(a) $\lim_{x\to\infty} \frac{2e^{-x} + \ln(x) + 5}{e^{-2x} + x^7} \underset{\text{Leading Behavior}}{=} \lim_{x\to\infty} \frac{\ln(x)}{x^7} = 0$

since x^7 grows faster than $\ln(x)$ as $x \to \infty$.

(b) $\lim_{x\to 0} \frac{e^x - 1}{\sin(4x)} \overset{\text{form } \frac{0}{0}}{\underset{\text{L'Hôpital's Rule}}{=}} \lim_{x\to 0} \frac{e^x}{4\cos(4x)} = \frac{e^0}{4\cos(4\cdot 0)} = \boxed{\frac{1}{4}}$

(c) $\lim_{x\to\infty} \frac{\ln(3x+1)}{\ln(10x^2+2)} \overset{\text{form } \frac{\infty}{\infty}}{\underset{\text{L'Hôpital's Rule}}{=}} \lim_{x\to\infty} \frac{\frac{1}{3x+1}(3)}{\frac{1}{10x^2+2}(20x)} = \lim_{x\to\infty} \frac{10x^2+2}{3x+1}\left(\frac{3}{20x}\right)$

$= \lim_{x\to\infty} \frac{30x^2+6}{60x^2+20x} \underset{\text{Leading Behavior}}{=} \lim_{x\to\infty} \frac{30x^2}{60x^2} = \lim_{x\to\infty} \frac{30}{60} = \boxed{\frac{1}{2}}$