Math 155 Exam 2 Fall 2010

1. (15 points) Compute the derivatives of the following functions. You do NOT need to simplify your answer. Use parentheses to indicate multiplication where appropriate, and make sure your notation is correct.

(a) $f(x) = 4x^2e^{-2x+1}$

$$\begin{aligned} f'(x) &= 4x^2(e^{-2x+1})(-2) + e^{-2x+1}(8x) \\ &= -8x^2e^{-2x+1} + 8xe^{-2x+1} \end{aligned}$$

(b) $g(x) = \dfrac{x^3}{\ln(x)}$

$$\begin{aligned} g'(x) &= \frac{\ln(x)(3x^2) - x^3(\frac{1}{x})}{[\ln(x)]^2} \\ &= \frac{3x^2\ln(x) - x^2}{[\ln(x)]^2} \end{aligned}$$

(c) $h(x) = \tan(x + 3\pi x^2)$

$$h'(x) = \sec^2(x + 3\pi x^2)(1 + 6\pi x)$$

(d) $m(x) = k(3x-2)^4 + \cos(4x-2)$ where k is a constant.

$$\begin{aligned} m'(x) &= 4k(3x-2)^3(3) - \sin(4x-2)(4) \\ &= 12k(3x-2)^3 - 4\sin(4x-2) \end{aligned}$$

(e) $p(x) = e^{\sin(3x)+2}$

$$\begin{aligned} p'(x) &= e^{\sin(3x)+2}(\cos(3x))(3) \\ &= 3\cos(3x)e^{\sin(3x)+2} \end{aligned}$$

2. (14 points) Consider the following discrete-time dynamical system:

$$x_{t+1} = x_t^2 + 7x_t + 8$$

(a) Find the equilibria algebraically.

$$\begin{aligned} x^* &= x^{*2} + 7x^* + 8 \\ 0 &= x^{*2} + 6x^* + 8 \\ 0 &= (x^* + 4)(x^* + 2) \\ x_1^* &= -4 \quad x_2^* = -2 \end{aligned}$$

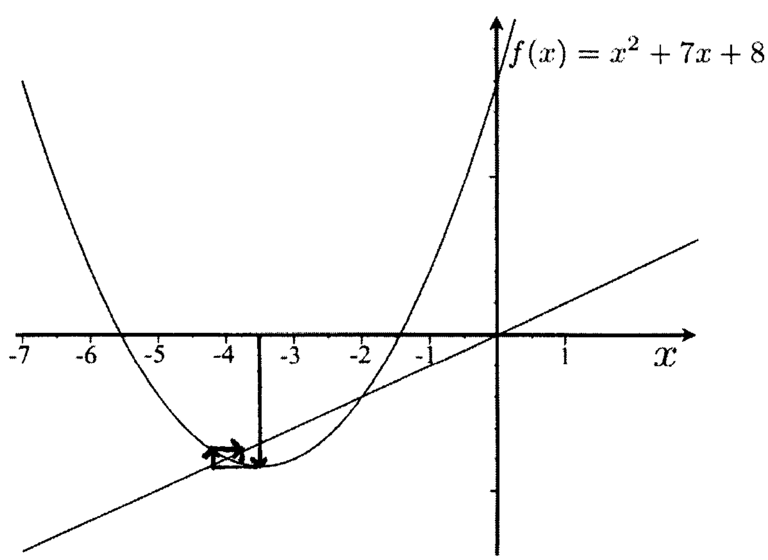
(b) Apply the Stability Test/Slope Criterion to each of the equilibria you found in (a). What can you conclude?

$$\begin{aligned} cf(x) &= x^2 + 7x + 8 \\ f'(x) &= 2x + 7 \end{aligned}$$

- $|f'(-2)| = |-4+7| = 3 > 1$. The Stability Test implies that $x_2^* = -2$ is unstable.
- $|f'(-4)| = |-8+7| = 1$. The Stability Test gives us no information about the stability of $x_1^* = -4$.

(c) The updating function is graphed below, along with the diagonal. Cobweb for at least 3 steps starting from the initial condition $x_0 = -3.5$. Describe in detail the long-term behavior of this solution.

The iterates oscillates about and approaches $x_1^* = -4$.



3. (15 points) Suppose that the population x_t of octopuses satisfies the discrete-time dynamical system

$$x_{t+1} = \frac{(p-5)x_t}{1+2x_t},$$

where p is a parameter.

(a) Verify algebraically that the equilibria are $x^* = 0$ and $x^* = \frac{p-6}{2}$.

$x^* = 0$: $\quad \frac{(p-5)(0)}{1+2(0)} = \frac{0}{1} = 0$ ✓

$x^* = \frac{p-6}{2}$: $\quad \frac{p-5\left(\frac{p-6}{2}\right)}{1+2\left(\frac{p-6}{2}\right)} = \frac{(p-5)\left(\frac{p-6}{2}\right)}{-5+p} = \frac{p-6}{2}$ ✓

(b) Show that the derivative of the updating function is $\frac{p-5}{(1+2x)^2}$.

$$f(x) = \frac{(p-5)x}{1+2x}$$

$$f'(x) = \frac{(1+2x)(p-5) - (p-5)x(2)}{(1+2x)^2} = \frac{p-5}{(1+2x)^2}.$$

(c) Use the Stability Criterion/Stability Theorem to find the range of p for which $x^* = 0$ is stable.

$|f'(0)| = \left|\frac{p-5}{(1+2(0))^2}\right| = |p-5|$;

0 is stable if $|p-5| < 1$;

$-1 < p-5 < 1$

$4 < p < 6$

(d) Use the Stability Criterion/Stability Theorem to find the range of p for which $x^* = \frac{p-6}{2}$ is *unstable*.

$\left|f'\left(\frac{p-6}{2}\right)\right| = \left|\frac{p-5}{\left(1+2\left(\frac{p-6}{2}\right)\right)^2}\right| = \left|\frac{p-5}{(p-5)^2}\right| = \frac{1}{|p-5|}$.

$\frac{p-6}{2}$ is unstable if $\left|\frac{1}{p-5}\right| > 1$; $\frac{1}{p-5} > 1$, or $\frac{1}{p-5} < -1$.

But, for $\frac{p-6}{2}$ to make sense as an equilibrium for this problem, $\frac{p-6}{2} \geq 0$, so $p \geq 6$, so that $\frac{1}{p-5} > 1$; $1 > p-5$; $p < 6$.

So, $x^* = \frac{p-6}{2}$ is <u>never unstable</u>.

4. (15 points) A diver jumps from a diving board. Her height (in feet) above the water at time t (in seconds) is given by $h(t) = -16t^2 + 16t + 32$, and she jumps at time $t = 0$.

(a) How high is the diving board from the water?

$h(0) = 32$ ft

(b) At what time t does the diver hit the water? (That is, at what time is the height equal to 0?)

$0 \stackrel{\text{set}}{=} h(t) = -16t^2 + 16t + 32$

$0 = -t^2 + t + 2$

$0 = t^2 - t - 2$

$0 = (t-2)(t+1)$

$t = 2$ or $t = -1$

$\boxed{t=2}$ time needs to be positive

(c) Find the velocity $v(t)$ and the acceleration $a(t)$ of the diver at time $t = 1$.

$v(t) = h'(t) = -32t + 16$. $v(1) = -16 \frac{\text{ft}}{\text{sec}}$

$a(t) = h''(t) = v'(t) = -32$ $a(1) = -32 \frac{\text{ft}}{\text{sec}^2}$

(d) At what time does the diver reach her maximum height above the water? What is this height?

Find critical points: $h'(t) = -32t + 16 \stackrel{\text{set}}{=} 0$

$32t = 16$

$\boxed{t = \frac{1}{2} \text{ sec.}}$

This is a local max since $h''(\frac{1}{2}) = -32 < 0$.

Also note that $h(\frac{1}{2}) = (-16)(\frac{1}{2})^2 + 16(\frac{1}{2}) + 32$

$\boxed{= 36 \text{ ft}}$ is the globally maximum height since $h(0) = 32 < 36$, and $h(2) = 0 < 36$.

5. (15 points) Consider the function $f(x) = x^3 - 30x^2 + 600$ on the interval $[-15, 35]$.

(a) Calculate $f'(x)$, and use this to find all the critical points of $f(x)$.

$f'(x) = 3x^2 - 60x$

critical points: $3x^2 - 60x \overset{\text{set}}{=} 0$

$x(3x - 60) = 0$

$\boxed{x = 0 \text{, and } x = 20}$

(b) Calculate $f''(x)$, and use this to find regions where $f(x)$ is concave up or concave down.

$f''(x) = 6x - 60$

$0 \doteq 6x - 60 \Rightarrow x = 10$

for $x < 10$

$f''(x) = 6x - 60 < 0$

$f''(x) < 0$ | 10 | $f''(x) > 0$ | x

$\boxed{\text{concave down on } (-\infty, 10)}$ $\boxed{\text{concave up on } (10, \infty)}$

(c) For each critical point, determine if $f(x)$ has a local maximum or a local minimum there. Justify your answer using the first or second derivative test.

2nd derivative test:

$x = 0$: $f''(0) = 6(0) - 60 < 0 \Rightarrow$ local max. at $x = 0$

$x = 20$: $f''(20) = 6(20) - 60 = 60 > 0 \Rightarrow$ local min. at $x = 20$

1st derivative test:

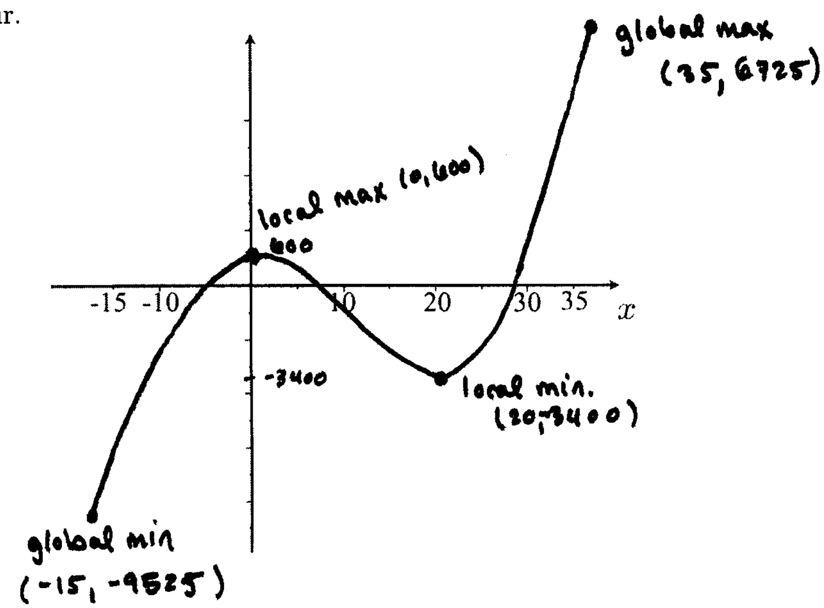
local. max at $x=0$; local min. at $x = 20$

+ + + | − − − − | + + +

0 | 20

$f'(x) > 0$ $f'(0) = 0$ $f'(x) < 0$ $f'(20) = 0$ $f'(x)$

(d) Use the information found above to sketch a graph of the function $f(x)$ on the interval $[-15, 35]$. Indicate where any local maxima, local minima, global maxima, or global minima occur.



6. (14 points) For the following functions, $f(x)$, find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$, and $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$.

(a) $f(x) = 17e^{-4x} + 4x^{-2} + 57x^{-8}$

$x \to \infty$: $\to 0 \quad \to 0 \quad \to 0$

$x \to 0$: $17 \quad \infty \quad \infty$

$f_\infty(x) = 4x^{-2}$

$f_0(x) = 57x^{-8}$

(b) $f(x) = \dfrac{13e^{-7x} + 17x^2 + x}{e^{3x} + x^2 - x}$

$top(x) = 13e^{-7x} + 17x^2 + x$ — $top_\infty(x) = 17x^2$, $top_0(x) = 13e^{-7x}$

$bot(x) = e^{3x} + x^2 - x$ — $bot_\infty(x) = e^{3x}$, $bot_0(x) = e^{3x}$

$\boxed{f_\infty(x) = \dfrac{top_\infty(x)}{bot_\infty(x)} = \dfrac{17x^2}{e^{3x}}}$.

$f_0(x) = \dfrac{top_0(x)}{bot_0(x)} = \dfrac{13e^{-7x}}{e^{3x}} = \boxed{13e^{-10x} = f_0(x)}$

(c) $f(x) = \dfrac{15x}{3+x}$

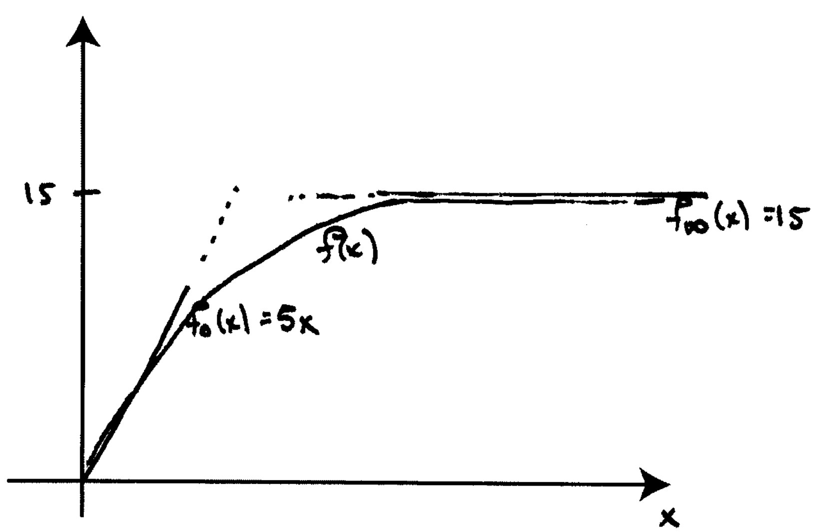
$top(x) = 15x$ — $top_\infty(x) = 15x$, $top_0(x) = 15x$

$bot(x) = 3 + x$ — $bot_\infty(x) = x$, $bot_0(x) = 3$

$f_\infty(x) = \dfrac{top_\infty(x)}{bot_\infty(x)} = \dfrac{15x}{x} = 15$

$f_0(x) = \dfrac{top_0(x)}{bot_0(x)} = \dfrac{15x}{3} = 5x$

(d) For the function in part (c), use the method of matched leading behaviors to sketch the graph of $f(x)$ for $x \geq 0$. Graph and indicate where you have graphed $f_\infty(x)$, $f_0(x)$, and $f(x)$.



7. (12 points) Evaluate the following limits. Show all of your work. If you use leading behavior, justify your answer by explaining all of your steps. If you use L'Hopital's Rule, justify why it can be applied each time you use it.

(a) $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x - 3} \underset{\text{form } \frac{0}{0}}{\overset{\text{L'Hôp}}{=}} \lim_{x\to 3} \dfrac{2x - 1}{1} = \dfrac{6-1}{1} = \boxed{5.}$

(b) $\lim_{x\to\infty} \dfrac{\ln(3x^2)}{x^2 + x} \underset{\text{form } \frac{\infty}{\infty}}{\overset{\text{L'Hôp}}{=}} \lim_{x\to\infty} \dfrac{\frac{1}{3x^2}\frac{d}{dx}(3x^2)}{2x+1} = \lim_{x\to\infty} \dfrac{\frac{6x}{3x^2}}{2x+1} = \lim_{x\to\infty} \dfrac{2}{2x^2+x} = \boxed{0}$

(c) $\lim_{x\to\infty} \dfrac{3e^{2x} + x^2 + 100}{\ln(x) + e^x} \underset{\text{L.B.}}{=} \lim_{x\to\infty} \dfrac{3e^{2x}}{e^x} = \lim_{x\to\infty} 3e^x = \boxed{\infty}$

(d) $\lim_{x\to 0} \dfrac{3e^{2x} + 1}{x + 1} = \dfrac{3e^{2(0)}+1}{0+1} = \dfrac{4}{1} = \boxed{4}$