FA08 Exam 2

1. (15 pts) Find the derivatives of the following functions. You do not have to simplify your answers. Be sure to use parentheses to indicate multiplication where appropriate.

a) $f(x) = x^4 - 5x^2 + \frac{x}{3} - \sqrt{\pi}$

$$f'(x) = 4x^3 - 10x + \frac{1}{3}$$

b) $g(\theta) = sin(\theta)cos(\theta)$

$$g'(\theta) = \sin(\theta)(-\sin\theta) + \cos(\theta)\cos(\theta)$$

c) $v(t) = \dfrac{t^3}{e^t - 1}$

$$v'(t) = \frac{(e^t-1)(3t^2) - t^3e^t}{(e^t-1)^2}$$

d) $h(x) = \ln(2x - 4)$

$$h'(x) = \frac{1}{2x-4}(2)$$

e) $f(x) = \sec(x^2)$

$$f'(x) = \sec(x^2)\tan(x^2)(2x)$$

2. (15 pts) Suppose a population of insects is governed by the updating function

$$x_{t+1} = \frac{x_t}{x_t - 1}.$$

A graph of this updating function is provided below.

a) Find the equilibria algebraically.

$$x^* = \frac{x^*}{x^* - 1}$$

$$x^*(x^* - 1) - x^* = 0$$

$$x^*(x^* - 1 - 1) = 0$$

$$\boxed{x^* = 0} \quad \text{and} \quad \boxed{x^* = 2}$$

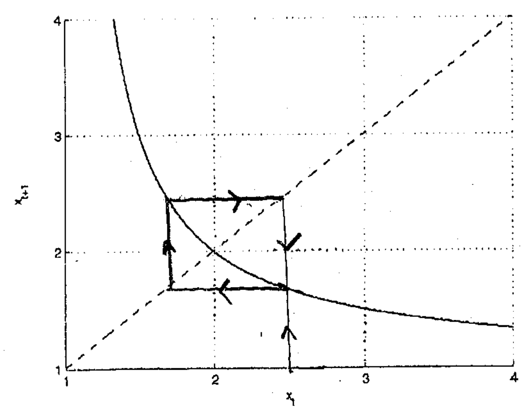
b) Find the slope of the updating function at the nonzero equilibrium.

$$f(x) = \frac{x}{x-1}$$

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$$

$$f'(2) = -\frac{1}{(2-1)^2} = \boxed{-1}$$

c) Cobweb from $x_0 = 2.5$ on the graph below, and describe (in detail) the behavior of the population. Assume each t represents a new generation of insect and t is in years.



The points $x_1 = 2.5$ and $x_2 = 1.667$ represent a 2-cycle. The population alternates between these two points each successive generation.

3. (12 pts) A ball is thrown upward from ground level so that it's height in meters after t seconds is given by

$$y(t) = 16.6t - 4.9t^2.$$

a) What is the velocity of the ball at time t?

$y'(t) = 16.6 - 9.8t$ m/s

b) What is the acceleration of the ball at time t?

$y''(t) = -9.8$ m/s^2

c) At what time will the ball hit the ground?

$0 = 16.6t - 4.9t^2$

$0 = t(16.6 - 4.9t)$

$0 = t$ or $16.6 = 4.9t$

$t \approx 3.3878$ seconds

4. (16 pts) Evaluate the following limits. Show all of your work. If you use L'Hospital's Rule, justify why it can be applied each time you use it. If you use knowledge of leading behaviors, justify your work.

a) $\lim_{x\to\infty} \frac{e^{3x}+7x}{x^2+4x-2}$

form $\frac{\infty}{\infty}$ L'Hôpital's

$\lim_{x\to\infty} \frac{3e^{3x}+7}{2x+4}$ form $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{9e^{3x}}{2} = \boxed{\infty}$

b) $\lim_{x\to 0} \frac{ln(x^2+1)}{x^2+3x}$

form $\frac{0}{0}$ L'Hôpital's

$\lim_{x\to 0} \frac{\frac{1}{x^2+1}(2x)}{2x+3} = \lim_{x\to 0} \frac{2x}{(x^2+1)(2x+3)} = \boxed{0}$

c) $\lim_{x\to\infty} \frac{x^2+4}{x^3}$

form $\frac{\infty}{\infty}$ L'Hôpital's

$\lim_{x\to\infty} \frac{2x}{3x^2}$ form $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{2}{6x} = \boxed{0}$

d) $\lim_{x\to 1} \frac{x^2-1}{2x-2}$

$\lim_{x\to 1} \frac{x^2-1}{2x-2} = \lim_{x\to 1} \frac{(x+1)(x-1)}{2(x-1)} = \lim_{x\to 1} \frac{x+1}{2} = \boxed{1}$

5. (12 pts) Consider a bacterial population x_t governed by the updating function

$$x_{t+1} = r(1 - 2x_t)x_t, \quad r > 0$$

a) Find all equilibria of the updating function.

$x^* = r(1 - 2x^*)x^*$

$x^* = 0$ is an equil.

$\frac{1}{r} = 1 - 2x^*$

$-2x^* = \frac{1}{r} - 1$

$x^* = -\frac{1}{2r} + \frac{1}{2}$

b) For $r = 0.8$ determine the stability of each of the equilibria.

$f(x) = rx(1 - 2x) = rx - 2rx^2$

$f'(x) = r - 4rx$

When $r = 0.8$, $f'(x) = .8 - 3.2x$

$|f'(0)| = |0.8| < 1$ So $x^* = 0$ is stable.

The other equil. is $x^* = \frac{-1}{2(.8)} + \frac{1}{2} = -.125$

$|f'(-.125)| = |.8 - 3.2(-.125)| = 1.2$, so $x^* = -.125$ is unstable

c) Give the values of r for which the nonzero equilibrium is stable.

Find r so that

$|f'(x^*)| = |r - 4rx^*| < 1$

$f'(\frac{1}{2} - \frac{1}{2r}) = r - 4r\left(\frac{1}{2} - \frac{1}{2r}\right) = r - 2r + 2 = 2 - r$

$|2 - r| < 1$

$-1 < 2 - r < 1$

$-2 \quad -2 \quad -2$

$-3 < -r < -1$

$1 < r < 3$

6. (18 pts) Given the function $f(x) = x^3 + 3x^2 - 9x + 4$ on the interval $(-\infty, \infty)$,

a) Find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$, and $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$

$$f_\infty(x) = x^3$$

$$f_0(x) = 4$$

b) Find the critical points of $f(x)$.

$$f'(x) = 3x^2 + 6x - 9 = 0$$

$$3(x^2 + 2x - 3) = 0$$

$$3(x+3)(x-1) = 0$$

$$x = -3, 1$$

c) Find the intervals where $f(x)$ is increasing and decreasing on $(-\infty, \infty)$.

$+ \; + \;|\; - \; - \; - \;|\; + \; +$

$\quad\;\; -3 \qquad\quad 1$

$f'(-4) = 15 > 0$

$f'(0) = -9 < 0$

$f'(2) = 15 > 0$

incr. on $(-\infty, -3) \cup (1, \infty)$

decr. on $(-3, 1)$

Problem 6 continued.

d) Find the intervals where $f(x)$ is concave up and concave down on $(-\infty, \infty)$.

$$f''(x) = 6x + 6 = 0$$
$$x = -1$$

$$f''(0) = 6 > 0$$
$$f''(-2) < 0$$

$- \; - \; - \mid + \; + \; +$

down -1 up

conc. down on $(-\infty, -1)$

conc. up on $(-1, \infty)$

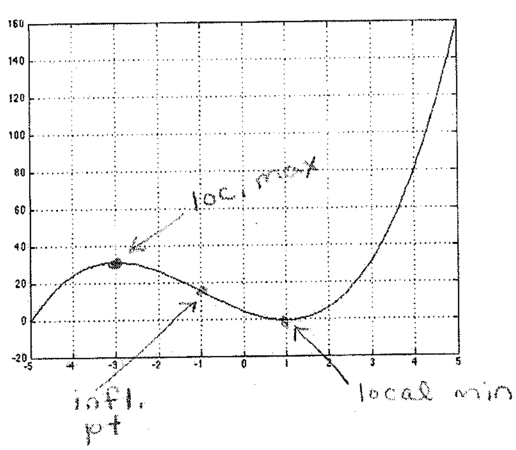
e) Identify the local minima. local maxima, and inflection points of $f(x)$ on $(-\infty, \infty)$.

local min at $x = 1$

local max at $x = -3$

infl. pt. at $x = -1$

f) Carefully sketch a graph of f, labeling the minima, maxima, and inflection points on the graph.



7. (12 pts) The thermic effect of food can be described for a particular individual by

$$F(t) = -10.28 + 176e^{-t/2}, \quad t \geq 0$$

where $F(t)$ is the thermic effect of food, measured in kJ/h and t is the number of hours that have elapsed since eating a meal.

a) Find the time after the meal when the thermic effect of food is maximized. Justify your answer with calculus.

$$F'(t) = -\frac{176}{2} e^{-t/2} < 0$$

The function is always decreasing, so the effect is maximal immediately after the meal.

b) Find the global maximum and global minimum on the interval $0 \leq t \leq 12$.

Since the func. is decreasing on $[0, 12]$ the global max is at $t = 0$

$F(0) = -10.28 + 176 =$

and the global min is at $t = 12$

$F(12) = -10.28 + 176e^{-6} =$