Math 155   Exam 2   Nov. 1, 5:15 pm   Fall 2007

NAME: _____

SECTION: _____ TIME: _____

INSTRUCTOR: _____

Instructions: The exam is closed book and closed notes. You may use an approved calculator, but be sure to show your work on each problem for full credit. Work that is crossed out or erased will not be graded. Turn in any scratch paper that you use during the exam. You will have one hour and 45 minutes to work on the exam.

Problem	Points	Score
1	12	
2	10	
3	20	
4	12	
5	16	
6	15	
7	15	
Total	100	

CONFIDENTIALITY PLEDGE

I agree that I will not share any information, either specific or general, about the problems on this examination with any other person until the exams have been returned to us in class.

_____
(Signature)

1. (12 pts) Evaluate the following limits. Show all of your work. If you use L'Hospital's Rule, justify why it can be applied each time you use it. If you use knowledge of leading behaviors, justify your work.

a) $\lim\limits_{x\to\infty} \dfrac{\sin x + 4x + 2}{x}$ $= \lim\limits_{x\to\infty} \left(\dfrac{\sin x}{x} + 4 + \dfrac{2}{x} \right)$

$$= \lim\limits_{x\to\infty} \dfrac{\sin x}{x} + \lim\limits_{x\to\infty} 4 + \lim\limits_{x\to\infty} \dfrac{2}{x}$$

$$= 0 + 4 + 0 = \boxed{4}$$

b) $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{x}$

form $\dfrac{0}{0}$ L'Hôpital's

$$\lim\limits_{x\to 0} \dfrac{\frac{1}{x+1}}{1} = \boxed{1}$$

c) $\lim\limits_{x\to 2} \dfrac{2 - 2e^{x-2}}{x-2}$ form $\dfrac{0}{0}$ L'Hôpital's

$$\lim\limits_{x\to 2} \dfrac{-2e^{x-2}}{1} = \boxed{-2}$$

d) $\lim\limits_{x\to\infty} (x^{-3} + 100x^{-1})$ $= \lim\limits_{x\to\infty} \left(\dfrac{1}{x^3} + \dfrac{1}{100x} \right) = \boxed{0}$

2. (10 pts) Given the function $f(x) = \frac{2x + e^{-x}}{x+1}$ on the interval $(-\infty, \infty)$

a) Find $f_\infty(x)$, the leading behavior of $f(x)$ as $x \to \infty$, and $f_0(x)$, the leading behavior of $f(x)$ as $x \to 0$.

$f_\infty(x) =$
$$\frac{2x}{x} = \boxed{2}$$

$$top_\infty(x) = 2x$$
$$bot_\infty(x) = x$$
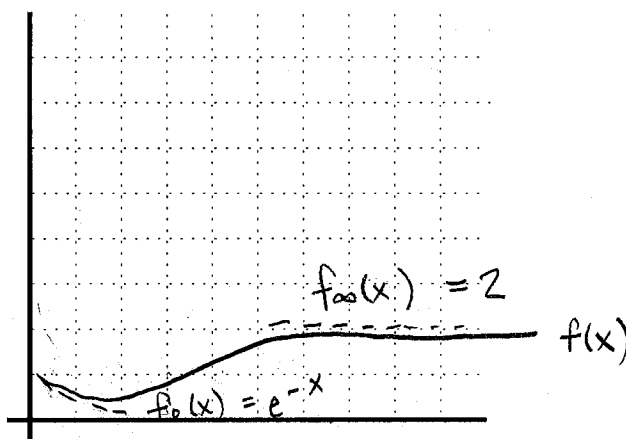
$$top_0(x) = e^{-x}$$
$$bot_0(x) = 1$$

$f_0(x) = \boxed{e^{-x}}$

b) Use the method of matched leading behaviors to sketch a graph of $f(x)$ on the interval $[0, \infty)$. Label your axes and indicate where you have graphed $f_0(x)$ and $f_\infty(x)$.

$$f_\infty(x) = 2$$
$$f(x)$$
$$f_0(x) = e^{-x}$$

3. (20 pts) Consider the function $f(x) = x + \cos x$ on $[0, 2\pi]$.

a) Find and classify the critical points. Find the intervals where $f(x)$ is increasing and decreasing on $[0, 2\pi]$.

$$f'(x) = 1 - \sin x \overset{set}{=} 0$$

$$\sin x = 1 \qquad x = \frac{\pi}{2}$$

$f'(x)$ $\quad \dfrac{\quad + \qquad + \quad}{\qquad \frac{\pi}{2}}$

so $f'(x)$ is increasing on all of $[0, 2\pi]$

b) Find the global minimum and global maxima of $f(x)$ on $[0, \pi]$.
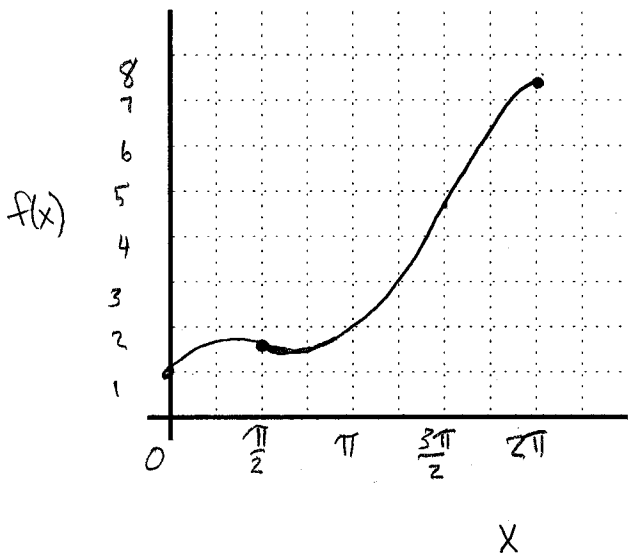
$$f(0) = 1$$

$$f(2\pi) = 2\pi + 1 \approx 7.2832$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$$

global min at $x = 0$

global max at $x = 2\pi$

c) Find and label the intervals where f is concave up and concave down (show your work), and sketch a graph of the function $f(x)$.

$$f''(x) = -\cos x \overset{set}{=} 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$f''(x)$ $\quad \dfrac{\quad - \qquad + \qquad - \quad}{\qquad \frac{\pi}{2} \qquad\quad \frac{3\pi}{2}}$

concave down: $\left[0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right]$

concave up: $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

4. (12 pts) The following updating function is a modification of the Ricker model for fisheries, where x_t represents the population size of the fish in the fishery for the tth generation.

$$x_{t+1} = \lambda x_t e^{\alpha - x_t^2}, \quad \alpha > 0, \quad \lambda > 0$$

a) Find all equilibria of the updating function.

$$X^* = \lambda X^* e^{\alpha - X^{*2}}$$

$$0 = \lambda X^* e^{\alpha - X^{*2}} - X^*$$

$$0 = X^* (\lambda e^{\alpha - X^{*2}} - 1)$$

$$\boxed{0 = X^*} \quad \text{and} \quad \lambda e^{\alpha - X^{*2}} = 1$$

$$e^{\alpha - X^{*2}} = \frac{1}{\lambda}$$

$$\alpha - X^{*2} = \ln(\frac{1}{\lambda})$$

$$\boxed{X^* = \sqrt{\alpha - \ln(\frac{1}{\lambda})}}$$

b) What is the physical meaning of each equilibrium? What additional condition on λ is needed to ensure each equilibrium makes sense?

$$X^* = 0 \quad \text{means there are no fish}$$

$$X^* = \sqrt{\alpha - \ln(\frac{1}{\lambda})} \quad \text{makes sense if}$$

$$\alpha - \ln(\frac{1}{\lambda}) > 0$$

$$\ln(\frac{1}{\lambda}) < \alpha$$

$$\frac{1}{\lambda} < e^{\alpha}$$

$$\boxed{\frac{1}{e^{\alpha}} < \lambda}$$

5. (16 pts) Find the derivatives of the following functions. You do not have to simplify your answers. Be sure to use parentheses to indicate multiplication where appropriate.

a) $f(x) = \tan x \cos 2x$.

$$f'(x) = \tan(x)(-\sin(2x))(2) + \cos(2x)\sec^2(x)$$

b) $f(y) = \frac{\sin(2y+1)}{y^2}$.

$$f'(y) = \frac{y^2(\cos(2y+1))(2) - \sin(2y+1)(2y)}{y^4}$$

c) $f(x) = \ln(\cos(2x^2 + \pi))$.

$$f'(x) = \frac{1}{\cos(2x^2+\pi)}(-\sin(2x^2+\pi))(4x)$$

d) $f(y) = e^{\sin y} \cos y$.

$$f'(y) = e^{\sin y}(-\sin y) + \cos y(e^{\sin y})(\cos y)$$

6. (15 pts) Suppose the position of an object attached to a spring is given by

$$p(t) = 2.0\pi + 2.0\cos(2\pi(t-3))$$

where t is time in sec, and $p(t)$ is distance in centimeters.

a. Find the velocity $v(t)$ and acceleration $a(t)$ as functions of t.

$$v(t) = p'(t) = -2\sin(2\pi(t-3))(2\pi)$$
$$= -4\pi\sin(2\pi(t-3))$$
$$a(t) = p''(t) = -2\cos(2\pi(t-3))(2\pi)(2\pi)$$
$$= -8\pi^2\cos(2\pi(t-3))$$

b. Give two times when the velocity is zero.

$$-4\pi\sin(2\pi(t-3)) \overset{set}{=} 0$$

$$2\pi(t-3) = 0 \qquad\qquad 2\pi(t-3) = \pi$$
$$2\pi t = 6\pi \qquad\qquad 2\pi t = 7\pi$$
$$\boxed{t = 3} \qquad\qquad \boxed{t = 3.5}$$

c. For what value of c is the following differential equation satisfied?

$$p''(t) + c(p(t) - 2.0\pi) = 0$$

$$-8\pi^2\cos(2\pi(t-3)) + c\left(2\pi + 2\cos(2\pi(t-3)) - 2\pi\right) = 0$$

$$-8\pi^2\cos(2\pi(t-3)) + 2c\cos(2\pi(t-3)) = 0$$

$$(-8\pi^2 + 2c)(\cos(2\pi(t-3))) = 0$$

$$-8\pi^2 + 2c = 0$$
$$2c = 8\pi^2$$
$$\boxed{c = 4\pi^2}$$

7. (15 points) Consider the following modified logistic growth model: $x_{t+1} = rx_t(1 - x_t^2)$

1. Let $f(x) = rx(1 - x^2)$, find $f'(x)$.

$$= rx - rx^3$$

$$\boxed{f'(x) = r - 3rx^2}$$

2. Evaluate $f'(x)$ at the equilibrium $x^* = \sqrt{1 - \frac{1}{r}}$.

$$f'\left(\sqrt{1-\tfrac{1}{r}}\right) = r - 3r\left(\sqrt{1-\tfrac{1}{r}}\right)^2 = r - 3r + 3r(\tfrac{1}{r}) = \boxed{-2r+3}$$

3. Let $r = 1.5$. Use the Stability Test to classify the nonzero equilibrium as stable, unstable or neither.

$$\left| -2(1.5) + 3 \right| = 0 < 1 \quad \text{stable}$$

4. Let $r = 2.6$. Use the Stability Test to classify the nonzero equilibrium as stable, unstable or neither.

$$\left| -2(2.6) + 3 \right| = |-2.2| = 2.2 > 1 \quad \text{unstable}$$

5. Below is the cobweb diagram starting from initial points $x_t = 0.8$ and $x_t = 6.7$. Based on the graph, what is the long-term behavior of the solution?
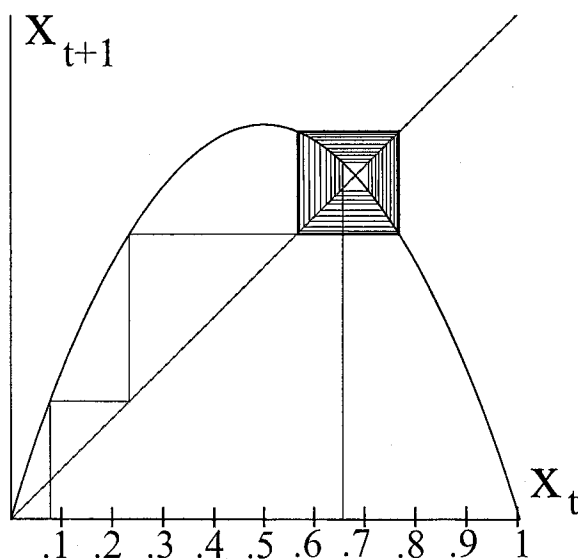
The long-term behavior is a 2-cycle.