Math 155
Exam 1
Spring 2013

1. (15 points) The following table gives data for the number c_{t+1} of chocolates in a desk at the end of week $t+1$ as a function of the number c_t of chocolates in the desk at the end of week t.

Number of chocolates at end of week t c_t	Number of chocolates at end of week $t+1$ c_{t+1}
100	30
120	40
200	80
250	105

(a) These data lie on a line–that is, c_{t+1} is a linear function of c_t. Find the formula for c_{t+1} as a function of c_t.

The line has slope $\frac{40-30}{120-100}\left(=\frac{80-40}{200-120}=\frac{105-80}{250-200}=\right)=\frac{1}{2}$.

(100, 30), for example, is a point on the line, which is therefore

$$c_{t+1}-30=\tfrac{1}{2}(c_t-100);$$

$$c_{t+1}=\tfrac{1}{2}c_t-50+30 \quad ; \quad \boxed{c_{t+1}=\tfrac{1}{2}c_t-20}$$

(b) Suppose that the number $C(t)$ of candies in the desk at time t obeys the equation

$$C(t)=C(0)e^{-0.05t}.$$

Find the half-life of the number of candies. For doubling time t_d,

$$\tfrac{1}{2}\cancel{C(0)}=\cancel{C(0)}e^{-0.05t_d}$$

$$\tfrac{1}{2}=e^{-0.05t_d}$$

$$\ln(1/2)=-0.05t_d;$$

$$\boxed{t_d=\frac{\ln(1/2)}{-0.05}\approx 13.86 \text{ days}}$$

(c) If the number of candies obeys the function $C(t)$ given in part (b), how long does it take for the number of candies to reduce to 20% of the number $C(0)$ of candies at time 0?

$$0.2\cancel{C(0)}=\cancel{C(0)}e^{-0.05t}$$

$$0.2=e^{-0.05t}$$

$$\ln(0.2)=-0.05t$$

$$\boxed{t=\frac{\ln(0.2)}{-0.05}\approx 32.18 \text{ days}}$$

2. (15 points) Two otters splash into their salt-water pool at the Zoo, spilling 67 L of water each time they splash. Their pool usually holds 5900 L of water. If one replaces the 67 L of water that they splash out with 67 L of water that has a concentration of 23 grams/L, the concentration of salt in the pool water may change.

(a) Fill in the blank boxes below to model the situation above. Let s_t represent the concentration of salt in the pool after the otters have splashed t times, measured in grams/L. Remember that concentration is equal to the amount of salt (grams) divided by the volume (L).

Step	Volume (L)	Total Salt (grams)	Salt Concentration (grams/L)
H_20 in pool before otters jump in	5900	$5900 s_t$	s_t
Water lost	67	$67 s_t$	s_t
H_20 in pool after otters jump in	5833	$5833 s_t$	s_t
Water replaced	67	1541	23
H_20 in pool after replacing water	5900	$5833 s_t + 1541$	$\frac{5833 s_t + 1541}{5900}$

(b) Write down the discrete-time dynamical system derived from the chart in part (a):

$$s_{t+1} = \frac{5833}{5900} s_t + \frac{1541}{5900}$$

$$(\simeq 0.9886 s_t + 0.2612)$$

(c) Suppose that 32 L of water at temperature T_1°C is mixed with 17 L of water at temperature T_2°C. Express the temperature of the resulting mixture in terms of a weighted average.

temperature of mixture

$$= \frac{32}{32+17} T_1 + \frac{17}{32+17} T_2$$

$$= \frac{32}{49} T_1 + \frac{17}{49} T_2 .$$

3. (14 points) (a) Write down a discrete-time dynamical system and an initial condition to describe the following situation: A lake is created at the zoo and stocked with 780 crayfish. Every year from then on, the otters catch 23% of the crayfish in the lake. The crayfish do not reproduce, but at the end of the year the lake is restocked by adding 540 crayfish. (Let C_t = the number of crayfish in the lake at the end of year t.)

$$C_0 = 780$$

$$C_{t+1} = (1 - 0.23)\,C_t + 540$$

$$= 0.77\,C_t + 540 \,.$$

(b) Find all equilibria of the discrete-time dynamical system

$$p_{t+1} = \frac{3p_t}{2p_t - k},$$

where k is a parameter. For what values of k is there a positive equilibrium?

$$p^* = \frac{3p^*}{2p^* - k}$$

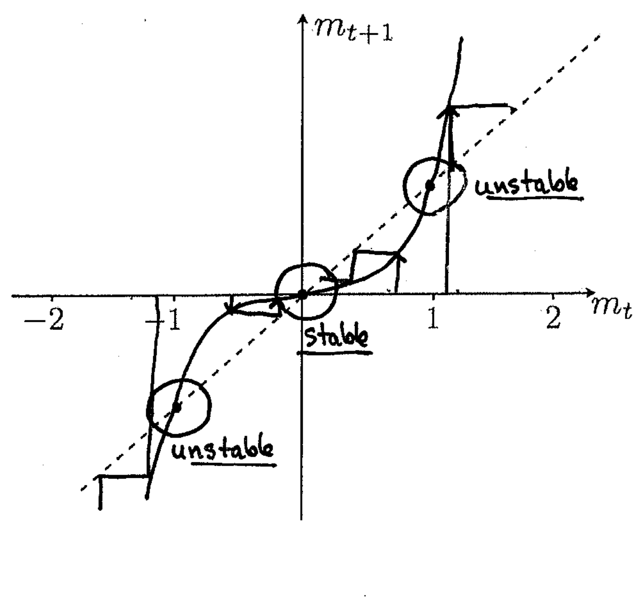
$$2(p^*)^2 - k p^* = 3p^*$$

$$2(p^*)^2 - k p^* - 3p^* = 0$$

$$p^*(2p^* - k - 3) = 0$$

$$\boxed{p^* = 0} \quad \text{or} \quad \boxed{p^* = \frac{k+3}{2}} \qquad \frac{k+3}{2} \text{ is positive if } k > -3$$

(c) Consider the discrete-time dynamical system $m_{t+1} = m_t^5$. i) *Graph* the updating function on the axes below (the diagonal $m_{t+1} = m_t$ is already graphed). ii) *Circle* ALL of the equilibria, and use *cobwebbing* to help you label each of the equilibria as stable or unstable.



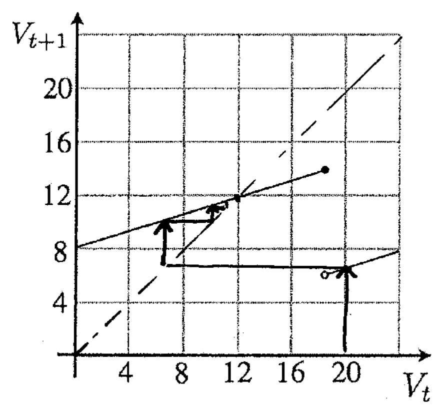
4. (14 points) Let V_t represent the voltage at the AV node in the heart model

$$V_{t+1} = \begin{cases} e^{-\alpha\tau} V_t + u, & \text{if } V_t \le e^{\alpha\tau} V_c \\ e^{-\alpha\tau} V_t, & \text{if } V_t > e^{\alpha\tau} V_c \end{cases}$$

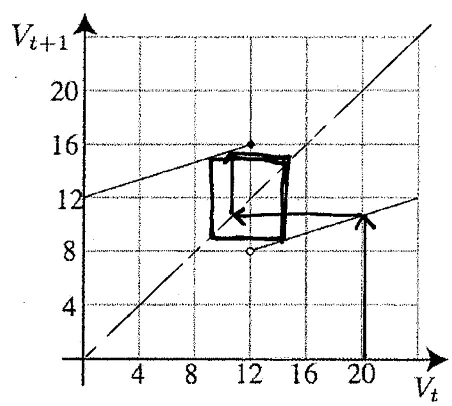
a) For each of the following two graphs of the updating function, cobweb for at least 4 steps starting from an initial value of $V_0 = 20$, and determine if the heart

i) is healthy, ii) has a 2:1 block, or iii) has the Wenckebach phenomenon.

Include arrows on your cobweb diagram.



Healthy



2:1 block

b) Now let $e^{-\alpha\tau} = 0.6$, $u = 5$, and $V_c = 20$. Does the system have an equilibrium? Justify your answer algebraically (that is, without drawing a graph), and find the equilibrium if there is one.

$$V_{t+1} = \begin{cases} 0.6 V_t + 5, & \text{if } V_t \le \frac{1}{0.6}(20) = \frac{10}{6}(20) = 33.\overline{3} \\ 0.6 V_t, & \text{if } V_t > \frac{1}{0.6}(20) = 33.\overline{3} \end{cases}$$

If the system has equilibrium V^*, then

$V^* = 0.6V^* + 5$;

$0.4V^* = 5$;

$V^* = \frac{5}{0.4} = 5\left(\frac{10}{4}\right) = \boxed{12.5}$

Since $12.5 < 33.\overline{3}$, yes, this system does have an equilibrium.

5. (14 points) (a) Find the following limits, if they exist. Show all of your work, and justify your answers to receive full credit. If a limit does not exist, write "DNE," and explain why it does not exist.

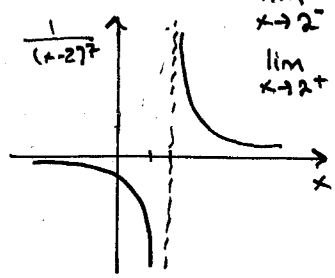
i) $\lim_{x\to 0} \frac{\tan x}{x+2} = \frac{\tan(0)}{0+2} = \boxed{0}.$

$\uparrow$ since $\frac{\tan x}{x+2}$ is continuous at $x=0$.

ii) $\lim_{s\to -6} \frac{s^2+3s-18}{s+6} = \lim_{s\to -6} \frac{(s+6)(s-3)}{s+6} = \lim_{s\to -6} (s-3) = \boxed{-9.}$

iii) $\lim_{x\to 2} \frac{1}{(x-2)^7}$

Graph:



$\lim_{x\to 2^-} \frac{1}{(x-2)^7} = -\infty$

$\lim_{x\to 2^+} \frac{1}{(x-2)^7} = \infty$

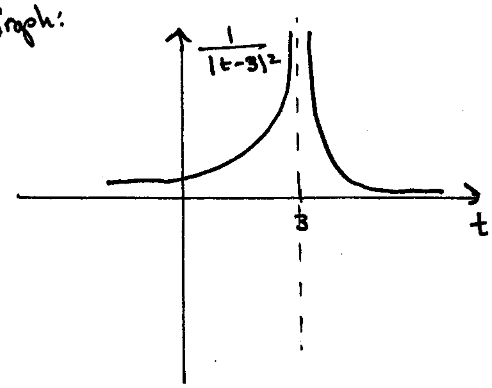
The left-hand limit does not equal the right-hand limit.

Hence,

$\boxed{\lim_{x\to 2} \frac{1}{(x-2)^7} \text{ DNE.}}$

iv) $\lim_{t\to 3} \frac{1}{(t-3)^2}$ $\boxed{= \infty}$

Graph:



6. (14 points) (a) We are interested in the density of a substance at a temperature of absolute zero (which is 0 Kelvin). However, we cannot measure the density directly at 0 Kelvin because it is impossible to reach absolute 0. Instead, we measure density for small values of the temperature.

Suppose that density D is a function of temperature T (measured in Kelvin) according to $D(T) = \frac{100}{1+2T}$.

i) What is $\lim_{T\to 0^+} \frac{100}{1+2T}$?

$$\lim_{T\to 0^+} \frac{100}{1+2T} = \frac{100}{1+2(0)} = \boxed{100}$$

ii) How close to 0 Kelvin would the temperature have to be for the density to be within 1% of the limit?

The density $D(T)$ is within 1% of the limit if

$$(0.99)(100) \le \frac{100}{1+2T} \le (100)(1.01);$$

$$99 \le \frac{100}{1+2T} \le 101$$

But, for any T, $\frac{100}{1+2T} \le 101$, so we only need to check $99 \le \frac{100}{1+2T}$;

$99(1+2T) \le 100$; $99 + 198T \le 100$; $198T \le 1$; $\boxed{T \le \frac{1}{198}}$

(b) Consider the function

$$f(x) = \begin{cases} \sin(x), & \text{if } x < \pi; \\ 0, & \text{if } x = \pi; \\ \cos(x) + 1 & \text{if } x > \pi. \end{cases}$$

Find the following limits, if they exist. If a limit does not exist, write "DNE," and explain why it does not exist.

$\lim_{x\to\pi^-} f(x) = 0$

$\lim_{x\to\pi^+} f(x) = 0$

$\lim_{x\to\pi} f(x) = 0$

Is this function continuous at $x = \pi$? Why or why not? Use the definition of continuity (*not* a phrase like "the graph can be drawn without lifting the pencil") to justify your answer.

Since $\lim_{x\to\pi} f(x) = 0 = f(\pi)$,

Yes f is continuous at π.

7. (14 points) Hildebrünn throws a ball up into the air from the top of a tower. Suppose that the height $h(t)$ (in meters) of the ball as a function of time t (in seconds) is given by $h(t) = -5t^2 + 30t + 20$.

(a) Find a formula for the slope of the secant line that passes through the points $(2, h(2))$ and $(2 + \Delta t, h(2 + \Delta t))$. Simplify your answer.

Slope of secant line $= \dfrac{h(2+\Delta t) - h(2)}{(2+\Delta t) - 2} = \dfrac{[-5(2+\Delta t)^2 + 30(2+\Delta t) + 20] - [-5\cdot 2^2 + 30\cdot 2 + 20]}{\Delta t}$

$= \dfrac{[-5(4 + 4\Delta t + \Delta t^2) + 60 + 30\Delta t + 20] - [-5\cdot 4 + 30\cdot 2 + 20]}{\Delta t}$

$= \dfrac{-20\Delta t \quad -5\Delta t^2 + 30\Delta t}{\Delta t} = \boxed{-5\Delta t + 10}$

(b) Find the average rate of change in h between time $t = 2$ and time $t = 2.5$.

$\Delta t = 2.5 - 2 = 0.5$.

Using the formula from (a),

the average rate of change $= -5(0.5) + 10 = -2.5 + 10$
$= \boxed{7.5}$

(c) Find the instantaneous rate of change of h at $t = 2$ *using the limit-definition of the derivative/instantaneous rate of change.* Is the height of the ball increasing or decreasing at time $t = 2$?

instantaneous rate of change $= \lim_{\Delta t \to 0}$ (average rate of change)

$= \lim_{\Delta t \to 0} (-5\Delta t + 10) = \boxed{10}$

part (a)

(d) Find the equation for the tangent line to the graph of $h(t)$ at the point $(2, h(2))$.

$(2, h(2)) = (2, 60)$

The slope of the tangent line is (from part (c)) 10.

$y - 60 = 10(t - 2)$

$y = 10t - 20 + 60$

$\boxed{y = 10t + 40}$