Math 155
Exam 1
Spring 2012

1. (15 points) Suppose that the mass M of an organism at time t is given by

$$M(t) = M_0 e^{0.8t},$$

where M_0 is the initial mass (in grams), and t is measured in months.

(a) Find the time it takes for the organism to double in mass.

$$2M_0 = M_0 e^{0.8t_d}$$
$$2 = e^{0.8t_d}$$
$$\ln(2) = \ln(e^{0.8t_d})$$
$$\ln(2) = 0.8t_d$$
$$t_d = \frac{\ln(2)}{0.8} \approx \underline{0.866}$$

(b) If the organism has a mass of 20 g at time $t = 5$, what was the mass of the organism at time $t = 0$? (That is, what is M_0?)

$$20 = M_0 e^{0.8(5)}$$
$$M_0 = \frac{20}{e^{0.8(5)}} \approx \underline{0.366}$$

(c) Consider the discrete-time dynamical system

$$w_{t+1} = 3.2w_t,$$

where w_t is the population of waterbears at time t, measured in hundreds. Find the (explicit) solution to the discrete-time dynamical system, given the inital condition $w_0 = 5$. Express your solution in terms of an exponential function in base e.

$$w_t = 5 \cdot 3.2^t = 5e^{\ln(3.2)t}$$
$$\approx \underline{5e^{1.163t}}$$

2. (15 points) Three walruses splash into their salt-water pool at the Zoo, spilling 80 gallons of water each time they splash. Their pool usually holds 600 gallons of water. If one replaces the 80 gallons of water that they splash with water that has a concentration of 3 mol/gallon, the concentration of salt may change.

(a) Fill in the blank boxes below to model the situation above. Let s_t represent the concentration of salt in the pool after the walruses have splashed t times, measured in mol/gallon. Remember that concentration is equal to the amount of salt (mol) divided by the volume (gal).

Step	Volume (gal)	Total Salt (mol)	Salt Concentration (mol/gal)
H_20 in pool before walruses jump in	600	$600 s_t$	s_t
Water lost	80	$80 s_t$	s_t
H_20 in pool after walruses jump in	520	$520 s_t$	s_t
Water replaced	80	$80(3) = 240$	3
H_20 in pool after replacing water	600	$520 s_t + 240$	$\frac{520 s_t + 240}{600}$

(b) Write down the discrete-time dynamical system derived from the chart in part (a):

$$s_{t+1} = \frac{520}{600} s_t + \frac{240}{600}$$

$$= \underline{\frac{13}{15} s_t + \frac{2}{5}}$$

(c) Suppose that 3.0 L of water with a salt concentration of C_1 moles/L is mixed with 5.0 L of water with a salt concentration of C_2 moles/L. Express the salt concentration of the resulting mixture in terms of a weighted average.

$$\frac{3}{3+5} C_1 + \frac{5}{3+5} C_2 = \underline{\frac{3}{8} C_1 + \frac{5}{8} C_2}$$

3. (14 points) (a) Write down a discrete-time dynamical system and an initial condition to describe the following situation: Each day, a patient uses up 35% of the medicine his bloodstream. However, he takes enough medicine at the end of each day to increase the concentration of medicine in his bloodstream by 4 milligrams per liter. The patient starts with a concentration of medicine in his bloodstream equal to 10 milligrams per liter. (Let M_t = the concentration of medicine on day t, in milligrams per liter.)

$$M_0 = 10$$

$$M_{t+1} = (1-0.35)M_t + 4$$
$$= 0.65M_t + 4$$

(b) Find all equilibria of the discrete-time dynamical system

$$z_{t+1} = \frac{kz_t}{3 - z_t},$$

where k is a parameter. For what values of k is there a positive equilibrium?

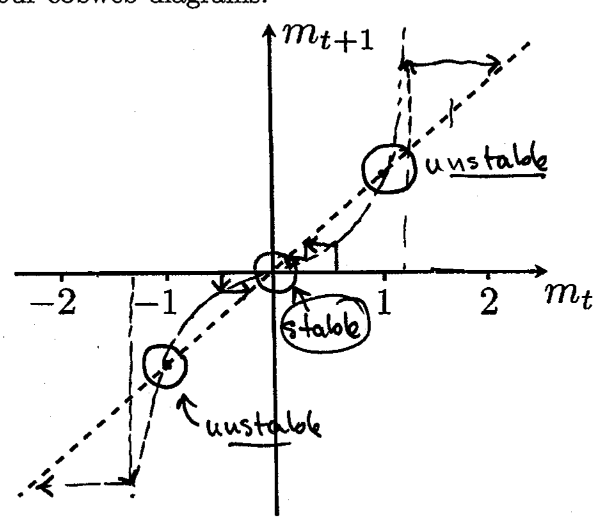
$$z^* = \frac{kz^*}{3-z^*}$$

$$z^*(3-z^*) = kz^*$$

$\boxed{z^* = 0}$, or $3 - z^* = k$; $\boxed{z^* = 3-k}$

There is a positive equilibrium for $3-k > 0$;
$k < 3$

(c) Consider the discrete-time dynamical system $m_{t+1} = m_t^3$. i) *Graph* the updating function on the axes below (the diagonal $m_{t+1} = m_t$ is already graphed). ii) *Circle* ALL of the equilibria, and use *cobwebbing* to help you label each of the equilibria as stable or unstable. Include arrows on your cobweb diagrams.



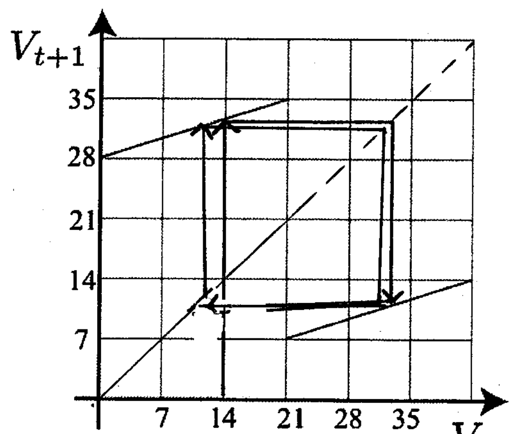
4. (14 points) Let V_t represent the voltage at the AV node in the heart model

$$V_{t+1} = \begin{cases} e^{-\alpha\tau}V_t + u, & \text{if } V_t \le e^{\alpha\tau}V_c \\ e^{-\alpha\tau}V_t, & \text{if } V_t > e^{\alpha\tau}V_c \end{cases}$$

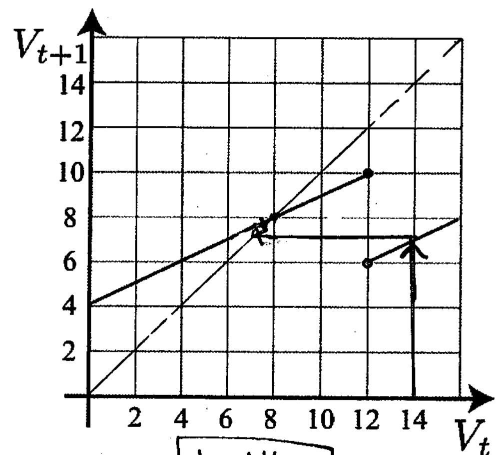
a) For each of the following two graphs of the updating function, cobweb starting from an initial value of $V_0 = 14$, and determine if the heart

i) is healthy, ii) has a 2:1 block, or iii) has the Wenckebach phenomenon.

Include arrows on your cobweb diagram.



2:1 block



healthy

b) Now let $e^{-\alpha\tau} = 0.75$, $u = 6$, and $V_c = 27$. Does the system have an equilibrium? Justify your answer algebraically (that is, without drawing a graph), and find the equilibrium if there is one.

$$V_{t+1} = \begin{cases} 0.75V_t + 6, & V_t \le \frac{1}{0.75}(27) = 36 \\ 0.75V_t, & V_t > 36 \end{cases}$$

If the system has an equilibrium V^*, then $V^* \le 36$, and

$V^* = 0.75V^* + 6$;

$0.25V^* = 6$

$V^* = 24 < 36$.

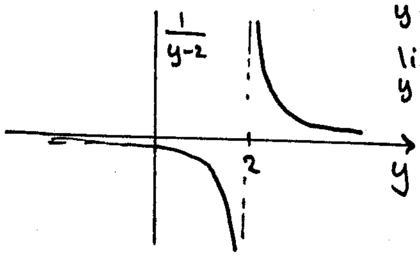
So, yes, $V^* = 24$ is an equilibrium for the system.

5. (14 points) (a) Find the following limits, if they exist. Show all of your work, and justify your answers to receive full credit. If a limit does not exist, write "DNE," and explain why it does not exist.

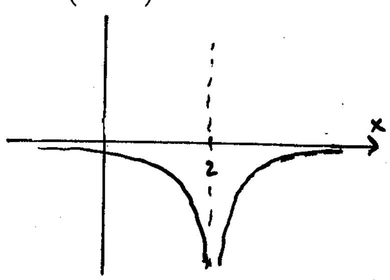
i) $\lim_{t \to 4} \frac{t^2 - 2t - 8}{t - 4}$ $= \lim_{t \to 4} \frac{(t-4)(t+2)}{t-4} = \lim_{t \to 4} (t+2) = 6.$

ii) $\lim_{y \to 2} \frac{1}{y - 2}$

$\left.\begin{array}{l} \lim_{y \to 2^-} \frac{1}{y-2} = -\infty \\ \lim_{y \to 2^+} \frac{1}{y-2} = \infty \end{array}\right]$ – not equal $\Rightarrow$ $\lim_{y \to 2} \frac{1}{y-2}$ DNE.



iii) $\lim_{x \to 2} \frac{-1}{(x - 2)^2}$ $= -\infty$.



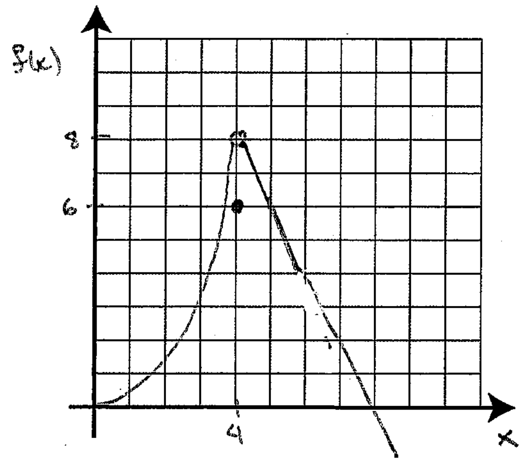
iv) $\lim_{\Delta x \to 0} \frac{4(x_0 + \Delta x)^2 - 4x_0^2}{\Delta x}$ (x_0 is a constant.)

$= \lim_{\Delta x \to 0} \frac{4(x_0^2 + 2x_0\Delta x + \Delta x^2) - 4x_0^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{4x_0^2 + 8x_0\Delta x + 4\Delta x^2 - 4x_0^2}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{8x_0\Delta x + 4\Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(8x_0 + 4\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} 8x_0 + 4\Delta x = \boxed{8x_0}$

6. (14 points) (a) Accurately graph the following piecewise function, remembering to label your axes.

$$f(x) = \begin{cases} \frac{1}{2}x^2, & \text{if } x < 4; \\ 6, & \text{if } x = 4; \\ -2x + 16 & \text{if } x > 4. \end{cases}$$



Find the following limits, if they exist. If a limit does not exist, write "DNE," and explain why it does not exist.

$\lim_{x \to 4^-} f(x) = 8$

$\lim_{x \to 4^+} f(x) = 8$

$\lim_{x \to 4} f(x) = 8$

Is this function continuous at $x = 4$? Why or why not? Use the definition of continuity (*not* a phrase like "the graph can be drawn without lifting the pencil") to justify your answer.

No, since $\lim_{x \to 4} f(x) = 8 \neq 6 = f(4)$.

(b) Consider $f(x) = 5x^2$. Note that $\lim_{x \to 0^+} 5x^2 = 0$. How close must the input x be to 0 to guarantee that the output $5x^2$ is within 0.01 of the limit? Give your answer to at least four decimal places.

We need $-0.01 < 5x^2 < 0.01$

$$\frac{-0.01}{5} < x^2 < \frac{0.01}{5}$$

$$x < \sqrt{\frac{0.01}{5}} \approx 0.044(7)\ldots$$

$$\approx 0.045$$

7. (14 points) Hildebrünn the Hedgehog rolls down a hill, and the distance that she has rolled as a function of time t is given by $f(t) = 3t^2 - t + 2$, where $f(t)$ is in meters, and t is in seconds.

(a) Find a formula for the slope of the secant line that passes through the points $(2, f(2))$ and $(2 + \Delta t, f(2 + \Delta t))$.

$$\text{slope of secant line} = \frac{f(2+\Delta t)-f(2)}{(2+\Delta t)-2} = \frac{[3(2+\Delta t)^2-(2+\Delta t)+2] - [3\cdot 2^2 - 2 + 2]}{(2+\Delta t)-2}$$

$$= \frac{3(4+4\Delta t+\Delta t^2) - 2 - \Delta t + 2 - 3\cdot 4\,(-2+2)}{\Delta t} = \frac{12\Delta t + 3\Delta t^2 - \Delta t}{\Delta t} = \frac{11\Delta t + 3\Delta t^2}{\Delta t}$$

$$= \boxed{11 + 3\Delta t}$$

(b) Find the average rate of change in f (that is, Hildebrünn's average speed) between time $t = 2$ and time $t = 2.5$.

$$\Delta t = 2.5 - 2 = 0.5.$$

$$\text{AROC} = \underset{(a)}{\underset{\uparrow}{11 + 3\Delta t}} = 11 + 3(0.5) = \boxed{12.5}$$

(c) Find the instantaneous rate of change of f at $t = 2$ *using the limit-definition of the derivative/instantaneous rate of change.*

$$\text{IROC} = \lim_{\Delta t \to 0} \text{AROC} = \lim_{\Delta t \to 0} (11 + 3\Delta t) = \boxed{11}$$

(d) Find the average rate of change of $f(t)$ between the times t_0 and $t_0 + \Delta t$.

$$\text{AROC} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \frac{[3(t_0+\Delta t)^2 - (t_0+\Delta t) + 2] - [3\cdot t_0^2 - t_0 + 2]}{\Delta t}$$

$$= \frac{[3(t_0^2 + 2t_0\Delta t + \Delta t^2) - t_0 - \Delta t + 2] - 3t_0^2 + t_0 - 2}{\Delta t}$$

$$= \frac{6t_0\Delta t + 3\Delta t^2 - \Delta t}{\Delta t} = \boxed{6t_0 - 1 + 3\Delta t}$$