Math155 Exam1 Spring2011

1. (12 pts)

   (a) The amount of water (W) retained in high-elevation, arid environments depends on the number of whitebark pine trees (T) according to $W(T) = \frac{2T}{1+T}$ The number of trees depends on the number of Clark's Nutcrackers (N) according to $T(N) = 4N$, as nutcrackers (a type of bird) are excellent seed dispersers. Find a function for the amount of water retained as a function of the number of Clark's Nutcrackers.

$$W(N) = W(T(N)) = \frac{2\,T(N)}{1+T(N)} = \frac{2(4N)}{1+4N} = \frac{8N}{1+4N}.$$

$$\boxed{W(N) = \frac{8N}{1+4N}}$$

   (b) Consider the following data points: (1,3), (3,12), (4,24). Do these data lie on a line? Justify your answer *without* giving a graph of the data.

Slope between (1,3) and (3,12): $\frac{12-3}{3-1} = \frac{9}{2}$

Slope between (3,12) and (4,24): $\frac{24-12}{4-3} = \frac{12}{1} = 12$

different slopes

$\Rightarrow$

<u>NO</u>, the data do <u>not</u> lie on a line.

   (c) Suppose that 6 L of water at temperature T_1 is mixed with 10 L of water at temperature T_2 . Express the temperature of the resulting mixture in terms of a weighted average.

temp. of mixture $= \frac{6}{6+10} T_1 + \frac{10}{6+10} T_2$

$= \frac{6}{16} T_1 + \frac{10}{16} T_2 = \underline{\frac{3}{8} T_1 + \frac{5}{8} T_2}$

2. (12 points) Consider the discrete-time dynamical system with updating function

$$v_{t+1} = 0.6v_t,$$

where v_t is the population of viruses in your body at time t, measured in thousands.

(a) Find the (explicit) solution to the discrete-time dynamical system, given the inital condition $v_0 = 10$. Express your solution in terms of an exponential function in base e.

$$v_t = 10(0.6)^t$$

$$= 10e^{\ln(0.6)^t} = 10^{t\ln(0.6)}$$

$$\simeq 10e^{-0.511t}$$

$$\boxed{v_t = 10e^{-0.511t}}$$

(b) Find the half-life of the virus population.

$$v_t = 10e^{t_h\ln(0.6)}$$

$$5 = 10e^{t_h\ln(0.6)}$$

$$\frac{1}{2} = e^{t_h\ln(0.6)}$$

$$\ln(1/2) = t_h\ln(0.6)$$

$$t_h = \frac{\ln(1/2)}{\ln(0.6)} \simeq 1.357 \quad ; \boxed{t_h \simeq 1.357}$$

(c) At what time will the virus population equal 3000?

Since v is measured in thousands, we need to find the time at which $v = 3$.

$$3 \doteq 10e^{t\ln(0.6)}$$

$$0.3 = e^{t\ln(0.6)}$$

$$\ln(0.3) = \ln(0.6)t$$

$$\frac{\ln(0.3)}{\ln(0.6)} = t$$

$$\boxed{t \simeq 2.357}$$

3. (10 points) Consider a population of wolves living in a national park.

This population lives within 5 square kilometers, and the wolves are evenly distributed throughout the park.

Every winter, 1.5 square km of their habitat is coverered in snow, killing any wolves living there and making that part of the park uninhabitable.

The following spring, the snow melts, opening up that part of the park again, and at the same time, 10 new wolves are born.

The chart below will help you find a discrete-time dynamical system for the density of wolves in year t.

(a) Finish the chart below to find a discrete-time dynamical system to model the situation described above. Let w_t represent the density (in number of wolves per km^2) of the wolf population in year t.

Step	Area (km^2)	Number of Wolves	Wolf Density $\left(\frac{\text{wolves}}{\text{km}^2}\right)$
Wolf range before the snow	5	$5w_t$	w_t
Wolf range covered by snow	1.5	$1.5w_t$	w_t
Wolf range after the snow	3.5	$3.5w_t$	w_t
Range uncovered as snow melts	1.5	0	0
Wolves born after the melt	N/A	10	N/A
Wolf range after the snow melts	5	$3.5w_t + 10$	$\frac{3.5w_t + 10}{5}$

(b) Write down the discrete-time dynamical system derived from the chart in part (a):

$$w_{t+1} = \frac{3.5}{5}w_t + \frac{10}{5} = 0.7w_t + 2$$

4. (13 points) (a) Write down a discrete-time dynamical system and an initial condition to describe the following situation: A population of bacteria triples every hour, but ten thousand bacteria are removed before reproduction each hour to be converted into valuable biological by-products. (Let b_t = the number of bacteria at hour t, measured in thousands.)

$$b_{t+1} = 3(b_t - 10)$$

(b) Find all equilibria of the discrete-time dynamical system

$$m_{t+1} = am_t^2 + bm_t,$$

where a and b are parameters. Are there any choices of a and b for which there is only one equilibrium?

$$m^* = a(m^*)^2 + bm^*$$

$$0 = a(m^*)^2 + bm^* - m^*$$

$$= m^*(am^* + b - 1)$$

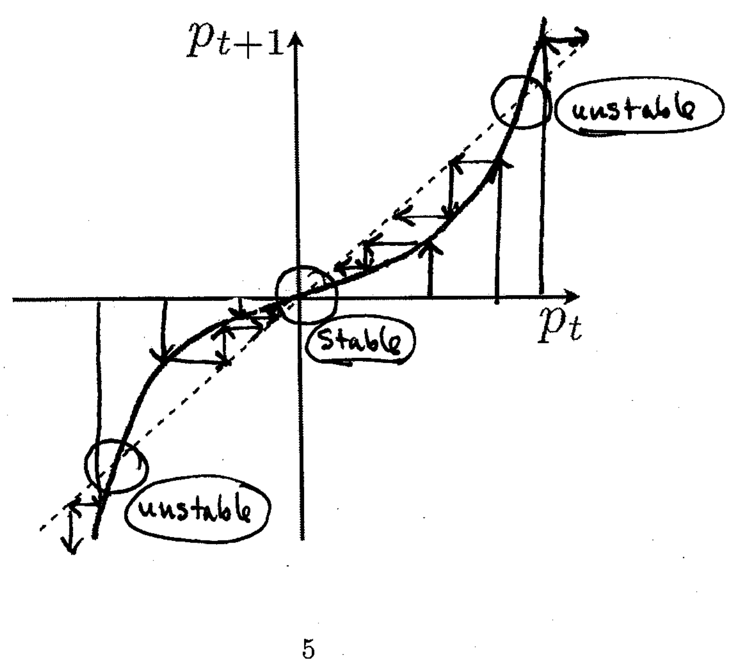
Either $m^* = 0$,

or

$am^* + b - 1 = 0 \rightarrow m^* = \frac{1-b}{a}$ if $a \neq 0$.

Equilibria: $\boxed{m^* = 0}$ and $\boxed{m^* = \frac{1-b}{a} \text{ if } a \neq 0}$ $\boxed{\text{Only 1 equilibrium } (m^* = 0) \text{ if } b = 1 \text{ (or } a = 0)}$

(c) The updating function for a discrete-time dynamical system $p_{t+1} = f(p_t)$ is graphed below, along with the diagonal. Circle each of the equilibria, and use *cobwebbing* to help you label each of the equilibria as stable or unstable. Include arrows on your cobweb diagrams.



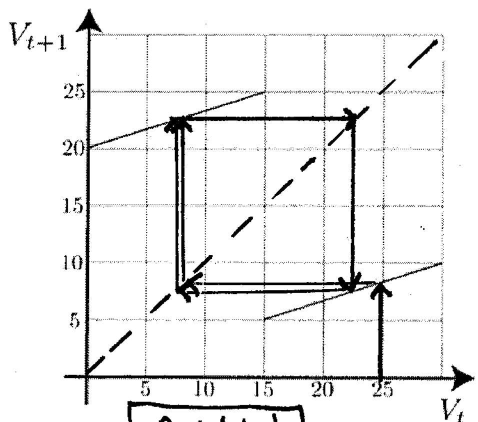
5. (13 points) Let V_t represent the voltage at the AV node in the heart model

$$V_{t+1} = \begin{cases} e^{-\alpha\tau}V_t + u, & \text{if } V_t \leq e^{\alpha\tau}V_c \\ e^{-\alpha\tau}V_t, & \text{if } V_t > e^{\alpha\tau}V_c \end{cases}$$

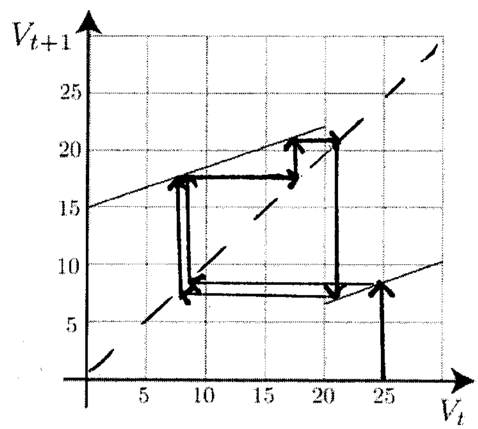
a) For each of the following two graphs of the updating function, cobweb starting from an initial value of $V_0 = 25$, and determine if the heart

i) is healthy, ii) has a 2:1 block, or iii) has the Wenckebach phenomenon.

Include arrows on your cobweb diagram.



$\boxed{\text{2:1 block}}$



$\boxed{\text{Wenckebach phenomenon}}$

b) Now let $e^{-\alpha\tau} = 0.8$, $u = 10$, and $V_c = 3$

i) Does the system have an equilibrium? Justify your answer (algebraically, not using a graph). If there is an equilibrium, find it.

If there is an equilibrium V^*, then it lies on the beat branch.

$V_{t+1} = e^{-\alpha\tau}V_t + u$;

$V^* = e^{-\alpha\tau}V^* + u = 0.8V^* + 10$;

$0.2V^* = 10$;

$V^* = 20$

But, $e^{\alpha\tau}V_c = \frac{1}{0.8}(3) = 3.75 < 20$. So, $\underset{V^*}{V^*} > e^{\alpha\tau}V_c$; the beat branch doesn't apply. So, $\boxed{\text{no equilibrium}}$

ii) Recall that $e^{-\alpha\tau}$ determines the decay in voltage at the AV node between signals from the SA node. If $\tau = 2$, determine the decay rate α. Round to three decimal places.

$e^{-\alpha\tau} = 0.8$

$\quad \parallel$

$e^{-2\alpha} \quad \Rightarrow \quad \ln(e^{-2\alpha}) = \ln(0.8)$

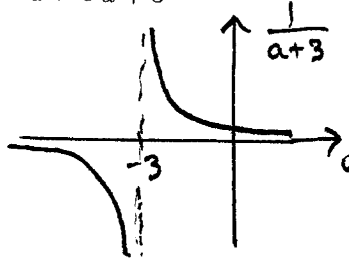
$-2\alpha = \ln(0.8)$

$$\boxed{\alpha = \frac{\ln(0.8)}{-2} \simeq \underline{0.112}}$$

6. (12 points) (a) Find the following limits, if they exist. Show all of your work, and justify your answers to receive full credit. If a limit does not exist, write "DNE," and explain why it does not exist.

i) $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x-2}$ $= \lim\limits_{x\to 2} \dfrac{(x+3)\cancel{(x-2)}}{\cancel{x-2}} = \lim\limits_{x\to 2}(x+3) = 2+3 \boxed{=5}$

ii) $\lim\limits_{a\to -3} \dfrac{1}{a+3}$



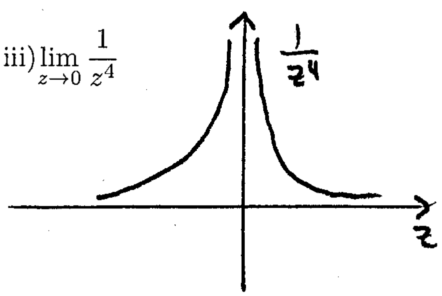
$\lim\limits_{a\to -3^-} \dfrac{1}{a+3} = -\infty$

but

$\lim\limits_{a\to -3^+} \dfrac{1}{a+3} = +\infty$

So, $\lim\limits_{a\to -3} \dfrac{1}{a+3}$ $\boxed{\text{DNE}}$

iii) $\lim\limits_{z\to 0} \dfrac{1}{z^4}$



$\lim\limits_{z\to 0^-} \dfrac{1}{z^4} = \infty$

$\lim\limits_{z\to 0^+} \dfrac{1}{z^4} = \infty$

; $\boxed{\lim\limits_{z\to 0} \dfrac{1}{z^4} = \infty}$

(b) Consider $f(x) = 4x^3$. Note that $\lim\limits_{x\to 2} 4x^3 = 32$. How close must the input x be to 2 to guarantee that the output $4x^3$ is within 0.01 of the limit? Give your answer to at least four decimal places.

We need

$32 - 0.01 < 4x^3 < 32 + 0.01$

$\dfrac{32-0.01}{4} < x^3 < \dfrac{32+0.01}{4}$

$\sqrt[3]{\dfrac{32-0.01}{4}} < x < \sqrt[3]{\dfrac{32+0.01}{4}}$

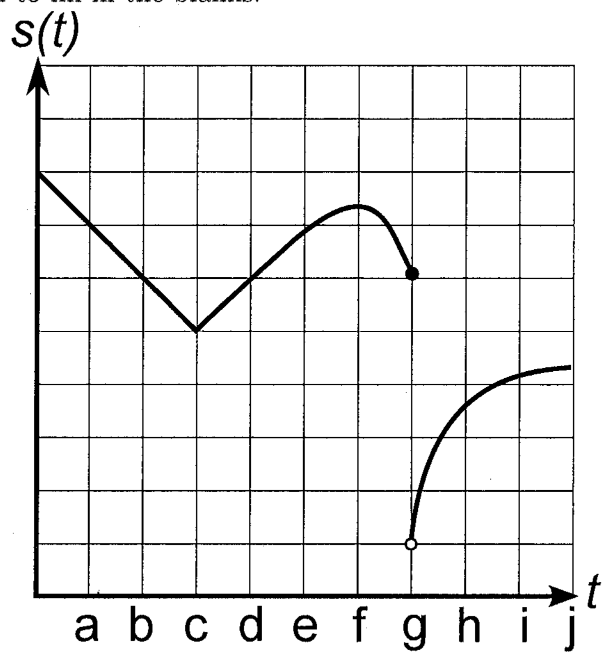
$1.99979 < x < 2.0002$

$2 - 1.99979 \approx 0.0002 = 2.0002 - 2$

$\boxed{\text{So, the input } \underline{x} \text{ must be within } 0.0002 \text{ of } 2.}$

7. (14 points)

(a) Use the graph to fill in the blanks.



i. List a value for t where s is discontinuous. g

ii. List a value for t where s is continuous but not differentiable. c

iii. List a value for t where $s'(t) = 0$. f

iv. List two values for t where $s'(t)$ does not exist. c and g

v. The largest value of $s'(t)$ is between the following two consecutive letters:
g $< x <$ h.

vi. $s(t)$ is decreasing the fastest between the following two consecutive letters:
f $< x <$ g.

(b) Consider the function

$$f(x) = \begin{cases} x+2, & \text{if } x < 3; \\ 6, & \text{if } x = 3; \\ 2x-1, & \text{if } x > 3. \end{cases}$$

i. $\lim_{x \to 3^-} f(x) =$ $\lim_{x \to 3^-} x+2 = 5$

ii. $\lim_{x \to 3^+} f(x) =$ $\lim_{x \to 3^+} 2x-1 = 5$

iii. Is $f(x)$ continuous at $x = 3$? Explain your answer fully using the definition of continuity.

$\lim_{x \to 3} f(x) = 5$, but $f(3) = 6 \neq 5$.

So, <u>NO</u>, f is <u>not</u> continuous at $x = 3$.

8. (14 points) Henriette the Hedgehog rolls down a hill, and the distance that she has rolled as a function of time t is given by $f(t) = t^2 + 3t + 5$, where $f(t)$ is in meters, and t is in seconds.

(a) Find a formula for the slope of the secant line that passes through the points $(2, f(2))$ and $(2 + \Delta t, f(2 + \Delta t))$.

slope of secant line =

$$\frac{f(2+\Delta t) - f(2)}{2+\Delta t - 2} = \frac{[(2+\Delta t)^2 + 3(2+\Delta t) + 5] - [2^2 + 3(2) + 5]}{\Delta t}$$

$$= \frac{4 + 4\Delta t + \Delta t^2 + 6 + 3\Delta t + 5 - [4 + 6 + 5]}{\Delta t} = \frac{4\Delta t + \Delta t^2 + 3\Delta t}{\Delta t} = \boxed{7 + \Delta t}$$

(b) Find the average rate of change in f (that is, Henriette's average speed) between time $t = 2$ and time $t = 2.5$.

The average rate of change is given by the slope of the secant line found in (a), with $\Delta t = 2.5 - 2 = 0.5$,

Average speed $= 7 + \Delta t = 7 + 0.5 = \boxed{7.5}$

(c) Find the instantaneous rate of change of f at $t = 2$ *using the limit-definition of the derivative/instantaneous rate of change.*

$$\lim_{\Delta t \to 0} (7 + \Delta t) = \boxed{7}$$

(d) Find the average rate of change of $f(t)$ between the times t_0 and $t_0 + \Delta t$. Simplify until you no longer have a fraction.

average rate of change

=

$$\frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \frac{[(t_0+\Delta t)^2 + 3(t_0+\Delta t) + 5] - [t_0^2 + 3t_0 + 5]}{\Delta t}$$

$$= \frac{\cancel{t_0^2} + 2t_0\Delta t + \Delta t^2 + \cancel{3t_0} + 3\Delta t + \cancel{5} - \cancel{t_0^2} - \cancel{3t_0} - \cancel{5}}{\Delta t}$$

$$= \frac{2t_0\Delta t + \Delta t^2 + 3\Delta t}{\Delta t} = \boxed{2t_0 + \Delta t + 3}$$