SP09 Exam 1

1. (12 pts) a) The number of mosquitos M that end up in a room is a function of how much the window is open (W sq. cm)

$$M(W) = 5W + 2$$

The number of bites B depends on the number of mosquitos according to

$$B(M) = 0.5M$$

Write B as a function of W.

$$B(M(W)) = 0.5(5W+2)$$

$$B(W) = 2.5W + 1$$

b) Most of the volume of an ant is contained in the head, the thorax, and the abdomen. Modeling the head and abdomen as spheres, and the thorax as a cylinder, suppose a certain ant has a head of radius 1.5 mm, an abdomen of radius 2.2 mm, and a thorax with a radius of 1.2 mm and length 2.7 mm.

Calculate the volume of the ant in square meters. Recall that the volume of a sphere is $\frac{4}{3}\pi r^3$.

Head: $1.5\,mm \cdot \frac{1\,m}{1000\,mm} = 0.0015\,m$

Volume: $\frac{4}{3}\pi(0.0015)^3 \approx 1.41 \cdot 10^{-8}\,m^3$

Abdomen: $2.2\,mm \cdot \frac{1\,m}{1000\,mm} = 0.0022\,m$

Volume: $\frac{4}{3}\pi(0.0022)^3 \approx 4.46 \cdot 10^{-8}\,m^3$

Thorax: $1.2\,mm \cdot \frac{1\,m}{1000\,mm} = 0.0012\,m$

$2.7\,mm \cdot \frac{1\,m}{1000\,mm} = 0.0027\,m$

Volume $\pi(0.0012)^2(0.0027) \approx 1.22 \cdot 10^{-8}\,m^3$

Total Volume: $\boxed{7.09 \cdot 10^{-8}\,m^3}$

2. (12 pts) Consider the discrete-time dynamical system with updating function

$$b_{t+1} = 0.4b_t$$

representing the population of bacteria in the t generation in thousands.

a) Find the solution to the discrete-time dynamical system b_t, given the initial condition $b_0 = 14$ thousand. Express your solution in terms of an exponential function in base e.

$b_0 = 14$ thousand $= (0.4)^0(14)$

$b_1 = 5.6 = (0.4)^1(14)$

$b_2 = 2.24 = (0.4)^2(14)$

$b(t) = 0.4^t(14)$ thousand

$b(t) = 14e^{(\ln .4)t} \approx 14e^{-.9163t}$ thousand

b) When with the population reach 7 thousand?

$7 = 14e^{-.9163t}$

$\frac{1}{2} = e^{-.9163t}$

$\frac{\ln(\frac{1}{2})}{-.9163} = t \approx .7565$ generations

3. (12 pts) a) Let M_t be a certain medication concentration in a bloodstream after t days, measured in milligrams per liter just after taking a daily dosage. Suppose 40% of the medication **remains** in the bloodstream after a day and a concentration of 18 milligrams per liter is added through the daily dosage. Write the updating function for this discrete-time dynamical system.

$$M_{t+1} = .4 M_t + 18$$

b) Find the equillibria of the following updating function algebraically.

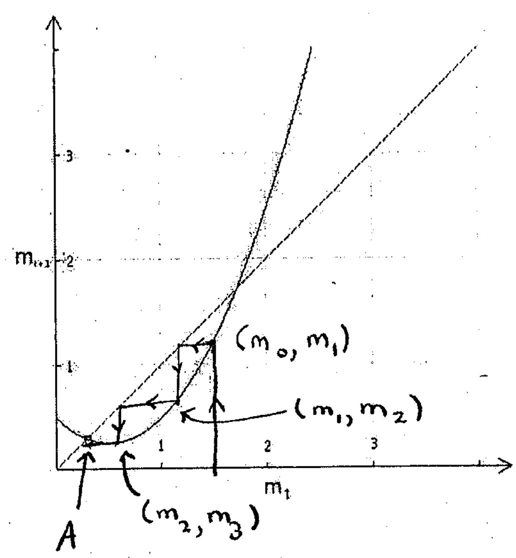
$$m_{t+1} = m_t^2 - m_t + 0.5.$$

$$m^* = m^{*2} - m^* + 0.5$$

$$m^{*2} - 2m^* + .5 = 0$$

$$m^* = \frac{2 \pm \sqrt{4 - 4(.5)}}{2} = 1 \pm \frac{\sqrt{2}}{2}$$

c) In the updating function graphed below, cobweb starting with an initial condition of $m_0 = 1.5$ and state the long term behavior for this initial value. Label m_1; m_2 and m_3 on your graph.

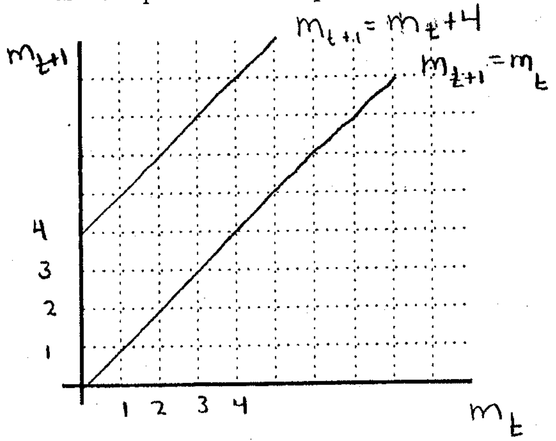


The long-term behavior is that the sol. approaches the equil. A.

4. (12 pts) a) For what values of α does an equilibrium exist for the following discrete time dynamical system? Find the equilibrium in terms of α. (Please show your work in the space to the right of the graph.)

$$m_{t+1} = \alpha m_t + 4$$

Graph the discrete time dynamical system on the axes below for a value of α for with there is no equilibrium. Explain.



$m^* = \alpha m^* + 4$

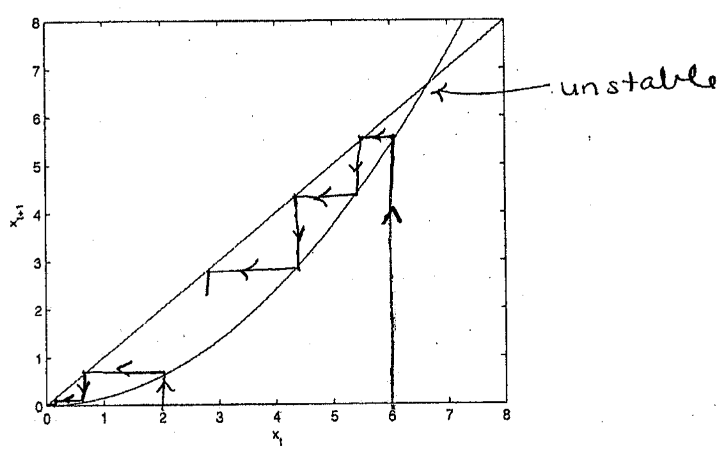
$(1-\alpha) m^* = 4$

$m^* = \dfrac{4}{1-\alpha}$

exists for all $\alpha \neq 1$

The graph of the updating func. is parallel to $m_{t+1} = m_t$ (the diagonal).

b) For the following graph of the discrete time dynamical system, use **cobwebbing** to help you label each point of equilibrium as **stable** or **unstable**.



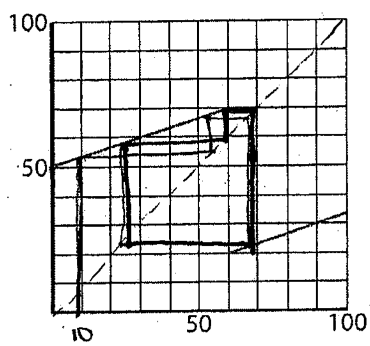
(0,0) is stable

5. (12 pts) Let V_t represent the voltage of the AV node in the Heart Model.

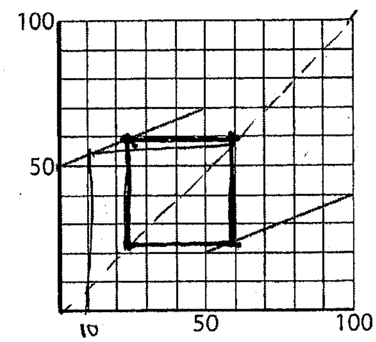
$$V_{t+1} = \begin{cases} e^{-\alpha\tau}V_t + u & \text{if } V_t \le e^{\alpha\tau}V_c \\ e^{-\alpha\tau}V_t & \text{if } V_t > e^{\alpha\tau}V_c \end{cases}$$

a) For each of the following two graphs of the updating function, cobweb starting from an initial value of $V_0 = 10$ and determine if the heart

i) is healthy, ii) has a 2:1 block, or iii) has the Wenckebach phenomenon.



Wenckebach



2 : 1 block

b) Now suppose that $e^{-\alpha\tau} = \frac{1}{3}$, $u = 20$, and $V_c = 20$.

i) If $V_0 = 10$, calculate V_1. Does the heart beat? Justify your answer.

$e^{\alpha\tau} V_c = (3)(20) = 60$

$V_0 = 10 < 60$, So yes, it beats.

So $V_1 = \frac{1}{3}(10) + 20 = 23.33$

ii) Does this system have an equilibrium? Justify your answer. If it has an equilibrium, find it algebraically.

$V^* = \frac{1}{3}V^* + 20$

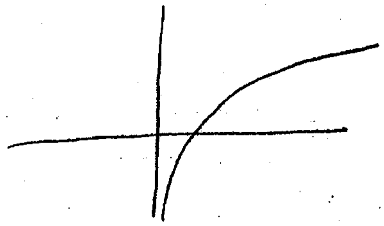
$\frac{2}{3}V^* = 20$

$V^* = \frac{(20)(3)}{2} = 30 < 60$

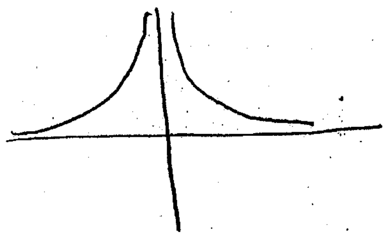
Yes $V^* = 30$ is the equil.

6. (12 pts) a) Calculate the following limits. Show all of your work to receive full credit.

i) $\lim_{x\to 0^+} \ln x$ $= -\infty$



ii) $\lim_{x\to 0} \frac{1}{x^2}$ $= \infty$



iii) $\lim_{x\to 3} \frac{x+3x^3}{x-2}$

$$= \frac{3+3(27)}{3-2} = 84$$

iv) $\lim_{x\to 1} \frac{x^2-4x+3}{x-1}$ $= \lim_{x\to 1} \frac{(x-3)(x-1)}{x-1} = \lim_{x\to 1} x-3 = 1-3 = -2$

b) Let $f(x) = \frac{2}{2x-6}$. Is $f(x)$ continuous at $x = 3$? Justify your answer.

No, it is not cont. at $x = 3$

$$\lim_{x\to 3^-} \frac{2}{2(x-3)} = -\infty$$

$$\lim_{x\to 3^+} \frac{2}{2(x-3)} = +\infty$$

So $\lim_{x\to 3} \frac{2}{2(x-3)}$ DNE

7. (14 pts) Three seals splash into their salt water pool at Sea World, spilling 20 gallons of water. Their pool usually holds 350 gallons. If one replaces the 20 gallons without adding salt, the concentration of salt will gradually decrease over time.

a) Fill in the four blank boxes in the chart below to model the situation described above. Let s_t represent the concentration of the salt.

Step	Volume	Total Chemical	Concentration
Before the seals jumped in	350 gal	350 s_t	s_t
Water lost	20 gal	20 s_t	s_t
After seals jumped in	330 gal	330 s_t	s_t
Pure water replaced	20 gal	0	0
After replacing with pure water	350	330 s_t	$\frac{330 s_t}{350}$

b) Write the discrete-time dynamical system: $s_{t+1} =$

$$s_{t+1} = \frac{330}{350} s_t \approx .94 s_t$$

c) The acceptable range of salt concentration for the seals is between 3 and 5 pph (parts per hundred). If you start with a concentration of 4 pph, how many times could 20 gallons of water be lost before one must add salt? Show your work.

$s_0 = 4$

$s_1 = (.94)(4) = 3.76$

$s_2 = 3.5344$

$s_3 = 3.322$

$s_4 = 3.123$

$s_5 = 2.94$

5 times

d) If the concentration of salt is currently at 2.98 pph and the pool has 330 gallons of water, how much water and what concentration would be needed to bring the pool back up to 350 gallons at 4 pph concentration of salt?

$$\frac{330(2.98)}{350} + \frac{20x}{350} = 4 \quad \text{(weighted average)}$$

$$330(2.98) + 20x = 1400$$

$$20x = 416.6$$

$$x = 20.83$$

20 gal at 20.83 pph

8. (14 pts) a) Find the slope of the secant line passing through the points $(t_0, f(t_0))$ and $(t_0 + \Delta t, f(t_0 + \Delta t))$ for the function $f(t) = 1.5t^2$.

$$\text{slope} = \text{ave r.o.c} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t}$$

$$= \frac{1.5(t_0+\Delta t)^2 - 1.5t_0^2}{\Delta t} = \frac{1.5(t_0^2 + 2t_0\Delta t + \Delta t^2) - 1.5t_0^2}{\Delta t}$$

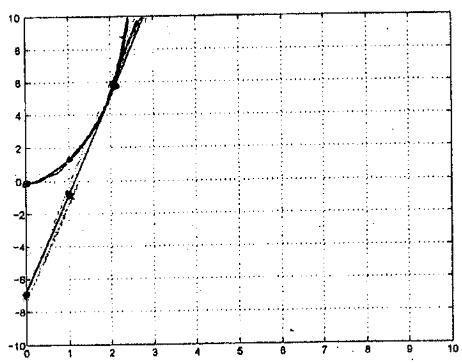
$$= \frac{3t_0\Delta t + 1.5\Delta t^2}{\Delta t} = 3t_0 + 1.5\Delta t$$

b) Find the equation of the secant line passing through the points $(t_0, f(t_0)) = (2.0, f(2.0))$ and $(t_1, f(t_1)) = (2.5, f(2.5))$. Graph the line on the plot below. Also graph $f(t)$ on the same set of axes. Label your graphs.

$t_0 = 2, \ \Delta t = .5$

Slope of secant line is

$3(2) + 1.5(.5) = 6 + .75 = 6.75$



$y - f(2) = 6.75(t-2)$

$y - 1.5(4) = 6.75t - 13.5$

$y - 6 = 6.75t - 13.5$

$y = 6.75t - 7.5$

at $t=1$:

$y = -.75$

at $t=2$ $\quad y = 6$

c) Find the instantaneous rate of change of $f(t)$ at time $t_0 = 2$.

$$\text{inst. r.o.c at time } t_0 = \lim_{\Delta t \to 0} (3t_0 + 1.5\Delta t) = 3t_0$$

At $t_0 = 2$, inst r.o.c $= 6$

or $\quad f'(t_0) = 3t_0$

$f'(2) = 6$