FA08 Exam 1

1. (12 pts) a) The length L of an insect in millimeters is a function of the temperature T (in degrees C) given by $L(T) = 20 + \frac{3T}{20}$. The volume V of the insect in mm^3 is a function of the length given by $V(L) = 2.3L^3$. Find the volume as a function of temperature. You do not have to simplify your answer.

$$V(T) = (V \circ L)(T) = 2.3\left(20 + \frac{3T}{20}\right)^3$$

b) What is the length and the volume of an insect that developed at 20 degrees C? (Be sure to provide units on your answers.)

$$L(20) = 20 + 3 = 23 \text{ mm}$$
$$V(23) = 2.3\,(23)^3 = 27984.1 \text{ mm}^3$$

c) A captive butterfly population produces three new butterflies per pair every six weeks, but 50 of the newly-hatched butterflies are removed to be sold to Butterfly Pavillions. Assuming that the adult butterflies die shortly before the new generation emerges, write a discrete-time dynamical system to describe the population.

Let b_t = # butterflies in t^{th} generation

$$b_{t+1} = \frac{3}{2} b_t - 50$$

2. (12 pts) Iodine 131, a radioactive isotope used to remove any remaining thyroid tissue after thyroid cancer surgery, has a half-life of 8.03 days and is modeled by $r(t) = r(0)e^{\alpha t}$ where $r(t)$ is measured in milliCuries.

a) Find α.

$$\frac{1}{2} = e^{\alpha(8.03)}$$

$$-\ln 2 = 8.03\alpha$$

$$\alpha = -.0863$$

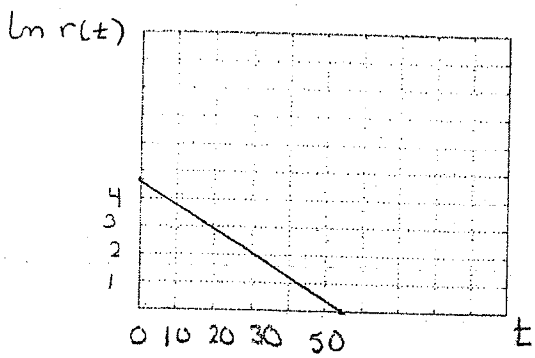
b) If a patient is given a dose of 100 milliCuries and radiation of 5 milliCuries or greater will continue to kill thyroid cells, how many days will pass before the initial Iodine 131 dose is no longer effective.

$$5 = 100e^{-.0863t}$$

$$\ln(0.05) = -.0863t$$

$$t = 34.71 \text{ days}$$

c) Plot $r(t)$ on a semilog graph given an initial dose of 100 milliCuries. Label your axes. Use increments of 10 days in t.



$$r(t) = 100e^{-.0863t}$$

$$\ln r = \ln 100 - .0863t$$

To find the t-intercept

$$0 = \ln 100 - .0863t_0$$

$$t_0 = 53.36$$

$$\ln r(0) = \ln 100 \approx 4.6$$

3. (12 pts) a) Find all non-negative equilibria for the following updating function.

$$y_{t+1} = \frac{y_t}{a - y_t}$$

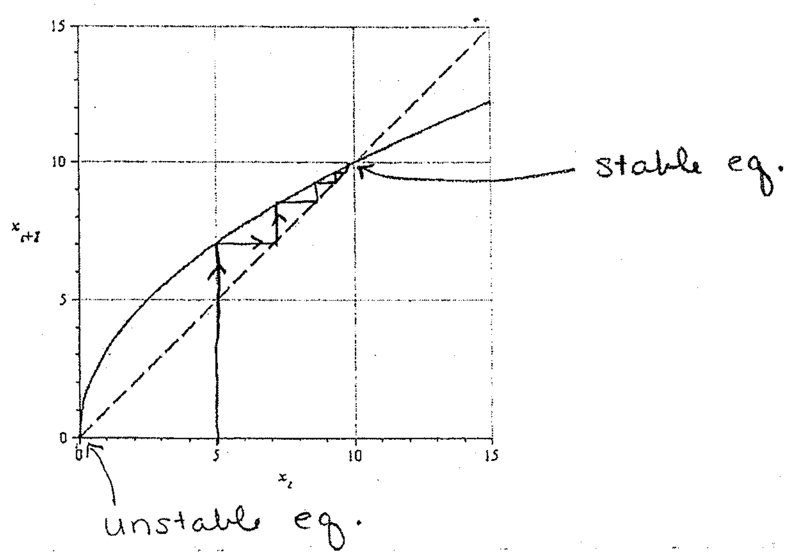
$$y^* = \frac{y^*}{a - y^*} \Rightarrow \boxed{y^* = 0} \text{ is an equil.}$$

$$y^*(a - y^*) = y^*$$

$$a - y^* = 1 \quad \text{for } y^* \neq 0$$

$$\boxed{y^* = a - 1}$$

b) For the following discrete time dynamical system, cobweb from an initial value of $x_0 = 5$, identify each point of equilibrium and indicate which are stable or unstable.



4. (12 pts) Suppose the function $b(t) = 2t^2 + 1$ represents the size of a bacterial population in millions at time t hours.

a) Find the average rate of change of the function $b(t)$ as a function of Δt. Simplify your answer.

$$\text{ave r.o.c} = \frac{b(t+\Delta t) - b(t)}{\Delta t}$$

$$= \frac{2(t+\Delta t)^2 + 1 - (2t^2+1)}{\Delta t}$$

$$= \frac{2(t^2 + 2t\Delta t + \Delta t^2 + 1) - 2t^2 - 1}{\Delta t} = \frac{4t\Delta t + 2\Delta t^2}{\Delta t}$$

$$= \boxed{4t + 2\Delta t}$$

b) How small must Δt be to get the average rate of change between $t = 1$ and $1 + \Delta t$ to be within 1% of 4 million?

1% of 4 million is $(.01)4 = .04$ million

$$4 - .04 < 4 + 2\Delta t < 4 + .04$$

$$-.04 < 2\Delta t < .04$$

$$-.02 < \Delta t < .02$$

c) Find the instantaneous rate of change of $b(t)$ at time t.

$$\lim_{\Delta t \to 0} 4t + 2\Delta t = 4t$$

5. (12 pts) a) Suppose 1.75 L of water at temperature T_1 is mixed with 2.5 L of water at temperature T_2. Express the temperature of the resulting mixture in terms of weighted averages.

Total amt of liquid is 4.25 L.

$\frac{1.75}{4.25} = .41 \qquad \frac{2.5}{4.25} = .59$

$\boxed{T = .41 T_1 + .59 T_2}$

b) Now suppose the mixture is as in part a, but before mixing, the temperature of each container of water is reduced by half its original temperature. What is the temperature of the resulting mixture, expressed as a weighted average?

$T = .41 \frac{T_1}{2} + .59 \frac{T_1}{2}$

c) What is the temperature in part a if $T_1 = 30°C$ and $T_2 = 56°C$?

$T = (.41)30 + .59(56)$

$= 12.3 + 33.04$

$= 45.34°C$

d) What is the temperature in part b if $T_1 = 30°C$ and $T_2 = 56°C$?

$T = \frac{1}{2}(45.34) = 22.67$

6. (12 pts) Let V_{t+1} represent the voltage of the AV node in the Heart Model.

$$V_{t+1} = \begin{cases} e^{-\alpha\tau} V_t + u & \text{if } V_t \leq e^{\alpha\tau}\, V_c \\ e^{-\alpha\tau} V_t & \text{if } V_t > e^{\alpha\tau}\, V_c \end{cases}$$

a.) Suppose $e^{-\alpha\tau} = 0.6065, V_c = 25, u = 8$, so that $e^{\alpha\tau} V_c \approx 41.2180$. Does this system have an equilibrium? Why or why not? Justify your answer. If it has an equilibrium, find it algebraically.

For $V_t < 41.2180$

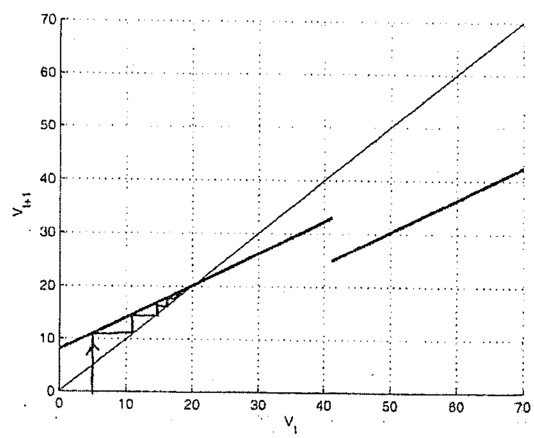
$V^* = .6065 V^* + 8$

$.3935 V^* = 8$

$V^* = 20.330$

Since $V^* < e^{\alpha\tau} V_c$, this is an equilibrium

b.) Cobweb from an initial value of $V_t = 5$. Diagnose this heart as healthy, having a 2:1 block, 3:1 block or the Wenkebach Phenomenon.



Eq. is stable, so this is a healthy heart.

7. (16 pts) Find each of the following limits algebraically. Show all of your work and/or justify your answers.

a) $\lim_{x\to 2} \frac{x^2+4x-12}{x^2-4}$ $= \frac{2^2+8-12}{4-4}$ Form $\frac{0}{0}$

$= \lim_{x\to 2} \frac{(x+6)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2} \frac{x+6}{x+2} = \frac{8}{4} = 2$

b) $\lim_{\Delta x\to 0} \frac{2(x^2+2x\Delta x+(\Delta x)^2)-2x^2}{\Delta x}$ $= \lim_{\Delta x\to 0} \frac{4x\Delta x + 2(\Delta x)^2}{\Delta x}$

$= \lim_{\Delta x\to 0} 4x + 2\Delta x = 4x$

c) $\lim_{x\to 3} \frac{3x^2+17x+10}{x+5}$ $= \frac{27+51+10}{8} = \frac{88}{8} = 11$

d) Given

$$c(t) = \begin{cases} 2t+2, & 0 \le t \le 2 \\ 5t - 4sin(\frac{\pi}{8}t), & 2 < t < 4 \\ 4t, & 4 \le t \le 6 \end{cases}$$

Find $\lim_{t\to 4^-} c(t)$.

$= \lim_{t\to 4} 5t - 4\sin(\frac{\pi}{8}t) = 20 - 4\sin\frac{\pi}{2} = 16$

Find $\lim_{t\to 4^+} c(t)$.

$= \lim_{t\to 4} 4t = 16$

8. (12 pts) Let $f(x) = 4x^2 + \frac{x}{2}$

a) Find the equation of the secant line to f that passes through $(1, f(1))$ and $(3, f(3))$.

$$f(1) = 4 + 1/2 = 4.5$$

$$f(3) = 36 + 3/2 = 37.5$$

$$\text{Slope } m = \frac{f(3)-f(1)}{3-1} = \frac{37.5 - 4.5}{2} = 16.5$$

$$y - f(1) = m(x-1)$$

$$y = 16.5(x-1) + 4.5$$

$$y = 16.5x - 12$$

b) Graph the secant line from part a on the axes below and graph the tangent line at $x = 1$ and label each line clearly.