Fall 2009
Exam 1

1. (11 points)

5 (a) The number of H1N1 viruses H (measured in trillions) that infect a college student is a function of the degree of immunosuppression I (the fraction of the immune system that is turned off by studying) according to $H(I) = 6I^2 + 2$. The fever F (measured in ° C) associated with an infection is a function of the number of viruses according to $F(H) = 37 + 0.5H$. Find fever as a function of immunosuppression. What will the fever be if immunosuppression is complete ($I = 1$)?

$H(I) = 6I^2+2$
$F(H) = 37 + 0.5H$

Find $F(I)$:

$F(H(I)) = F(6I^2+2) = 37 + 0.5(6I^2+2) = 37 + 3I^2 + 1$
$= 38 + 3I^2;$

If $I = 1$ $F(1) = 38 + 3 = 41$ $\boxed{F(I) = 38 + 3I^2}$

(b) Consider the data in the following table describing the number of students in Dorm Q that come down with the H1N1 flu as a function of the average number of hours that students in the dorm spend studying each day.

X: average number of hours per day spent studying	Y: number of students sick with H1N1
1	34
4.5	13
5	10

4 i. These data lie on a line (to convince yourself of this, you may graph the data, but you do **not** need to show a graph here). Find the equation of the line connecting the first two points.

$y - y_0 = m(x - x_0)$. $(x_0, y_0) = (1, 34)$

$m = \frac{13-34}{4.5-1} = \frac{-21}{3.5} = -6$

equation: $y - 34 = -6(x-1)$; $\underline{y = -6x + 40}$

2 ii. How many students does your equation predict would get sick in in Dorm Q if the average numbers of hours that students in this dorm spend studying each day is 8? Does this make sense? Why or why not?

$y = -6(8) + 40 = -8 < 0$

No, this does NOT make sense since students don't come in negatives.

2. (12 points) Consider the discrete-time dynamical system with updating function

$$b_{t+1} = 0.4 b_t$$

representing the population of bacteria.

5 (a) Find the solution of the discrete-time dynamical system if $b_0 = 1.0 \times 10^4$.

$$b_t = 1.0 \times 10^4 (0.4)^t$$

$$\left(= 1.0 \times 10^4 e^{\ln(0.4) t} \right)$$

7 (b) Find the half-life of the population.

$$\frac{1}{2}\left(\cancel{1.0 \times 10^4}\right) = \cancel{1.0 \times 10^4}(0.4)^t$$

$$\frac{1}{2} = 0.4^t$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(0.4^t\right)$$

$$\ln\left(\frac{1}{2}\right) = t \ln(0.4);$$

$$t = \frac{\ln(1/2)}{\ln(0.4)}$$

$$\simeq 0.756$$

or:

$$\frac{1}{2} \cancel{b_0} = \cancel{b_0} e^{\ln(0.4) t}$$

$$\frac{1}{2} = e^{\ln(0.4) t}$$

$$\ln(1/2) = \ln(0.4) t;$$

$$t = \frac{\ln(1/2)}{\ln(0.4)}$$

3. (14 points) Suppose that 40 percent of the medication in a patient's body is **removed** each day by the body but the patient takes an additional dose of 2 mg/L at the end of each day.

(a) Write the updating function $M_{t+1} = f(M_t)$ for the concentration of medicine on day $t+1$ as a function of the concentration on day t. Recall that

new concentration = old concentration - fraction used × old concentration + supplement.

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$$M_{t+1} = M_t - 0.4M_t + 2 = 0.6M_t + 2$$

$$\boxed{M_{t+1} = 0.6M_t + 2}$$

(b) Find the equilibrium point algebraically, if it exists.

4

$$M^* \doteq 0.6M^* + 2$$

$$0.4M^* = 2$$

$$M^* = 2/0.4 = 5$$

(c) Draw the graph of the updating function, identify the equilibrium if it exists, and cobweb the concentration for days 1,2, and 3 if the initial dose is $M_0 = 10$ mg/L. (Make sure to label clearly all points (M_t, M_{t+1}) on the updating function during cobwebbing.)

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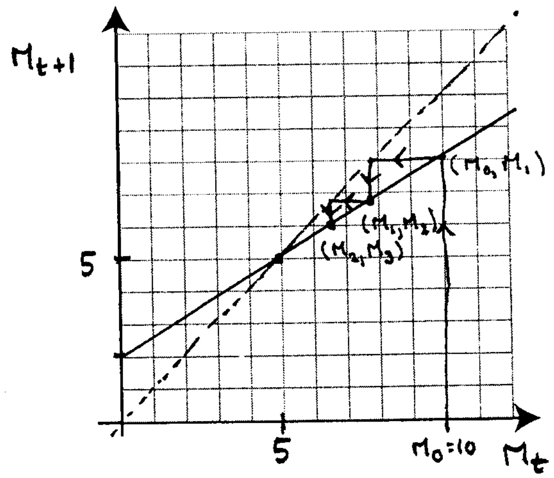

(d) If $M_0 = 10$, the solution to this discrete time dynamical system is $M_t = 5 \cdot (1 + (\frac{3}{5})^t)$ where t is time, measured in days. While the patient requires a high initial dose the patient will be at risk if the concentration remains above 6 mg/L for 3 days or more. Use the solution given to determine if this patient will be at risk. (*Hint:* What is M_3?)

2

$$M_3 = 5(1 + (3/5)^3) = 6.08 > 3.$$

Yes, The patient is indeed at risk.

4. (12 points)

(a) Find all non-negative equilibria of the discrete-time dynamical system

$$m_{t+1} = \frac{2m_t}{1+am_t},$$

where a is a positive parameter. How many equilibria are there if $a = 0$?

$a > 0$: $m^* \doteq \frac{2m^*}{1+am^*}$

$m^*(1+am^*) = 2m^*$

$m^* + a(m^*)^2 = 2m^*$

$a(m^*)^2 - m^* = 0$

$m^*(am^* - 1) = 0$

$m^* = 0$ or

$am^* - 1 = 0$; $m^* = \frac{1}{a}$.

equilibria: $\boxed{m^* = 0 \text{ and } m^* = \frac{1}{a}}$

If $a = 0$,

$m^* = \frac{2 \cdot m^*}{1}$;

$m^* = 0$ is the only equilibrium.

$\boxed{1}$

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(b) The updating function for a discrete-time dynamical system $x_{t+1} = f(x_t)$ is graphed below. Use *cobwebbing* to help you label each of the three equilibria as stable or unstable.

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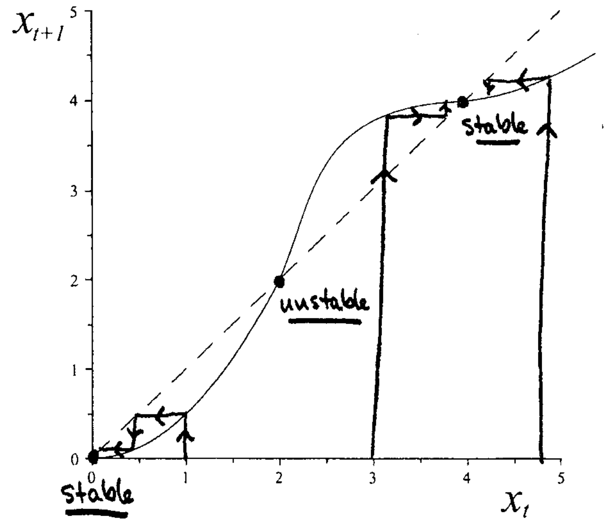

5. (12 points) Let V_{t+1} represent the voltage of the AV node in the heart model.

$$V_{t+1} = \begin{cases} e^{-\alpha\tau}V_t + u, & \text{if } V_t \leq e^{\alpha\tau}V_c \\ e^{-\alpha\tau}V_t, & \text{if } V_t > e^{\alpha\tau}V_c \end{cases}$$

(a) For each of the following two graphs of the updating function, cobweb starting from an initial value of $V_0 = 10$ and determine if the heart is healthy, has 2:1 AV block, or the Wenckebach phenomenon.

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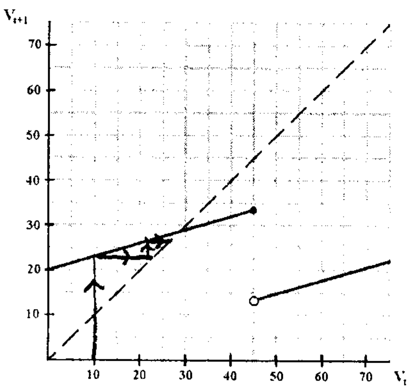
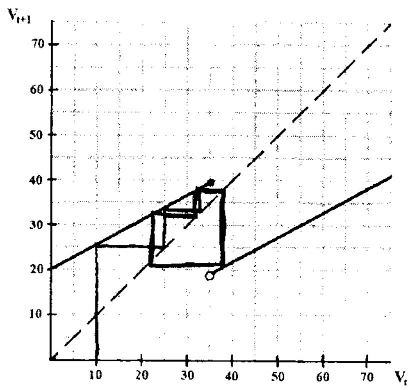


Healthy



more beats than skips;

Wenckebach phenomenon

(b) Let $e^{-\alpha\tau} = 0.6$, $V_c = 21$, and $u = 8$. If $V_0 = 26$ will the heart beat? Why or why not? Calculate V_1.

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$$e^{\alpha\tau}V_c = \frac{21}{0.6} = 35.$$

Since $V_0 = 26 < 35$, yes the heart will beat.

$$V_1 = 0.6(26) + 8 = \boxed{23.6 = V_1}$$

(c) Does the system described in part b have an equilibrium? Why or why not? If it has an equilibrium, find it algebraically.

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If there is an equilibrium V^* then

$$V^* = e^{-\alpha\tau}V^* + u = 0.6V^* + 8;$$

$$0.4V^* = 8;$$

$$V^* = 20$$

Since $V^* = 20 < e^{\alpha\tau}V_c = \frac{21}{0.6}$, V^* is indeed an equilibrium; yes, the system has an equilibrium.

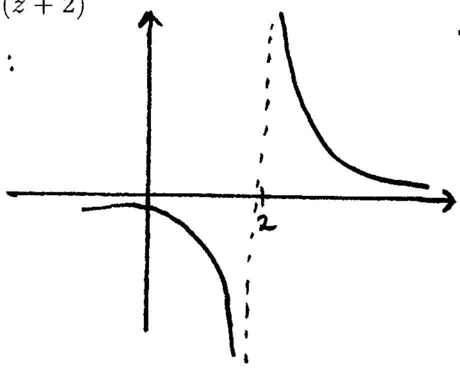
6. (13 points) (a) Find the following limits, if they exist. Show all of your work and justify your answers to receive full credit. If a limit does not exist, write "DNE," and explain why it does not exist.

2 i) $\lim_{x\to 1} \frac{7x^2+3x}{2x-1}$ $= \frac{7(1)^2+3(1)}{2(1)-1} = \underline{10}$

2 ii) $\lim_{t\to -2} \frac{t^2+9t+14}{t+2}$ $= \lim_{t\to -2} \frac{\cancel{(t+2)}(t+7)}{\cancel{t+2}} = \lim_{t\to -2} (t+7) = \underline{5}.$

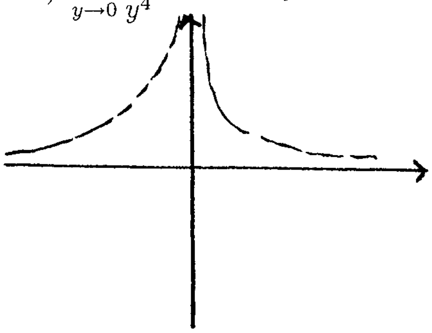
2 iii) $\lim_{z\to 2} \frac{z}{(z-2)(z+2)}$ DNE

Graph:



DNE since $\lim_{z\to 2^+} \frac{z}{(z-2)(z+2)} = +\infty \neq \lim_{z\to 2^-} \frac{z}{(z-2)(z+2)} = -\infty$.

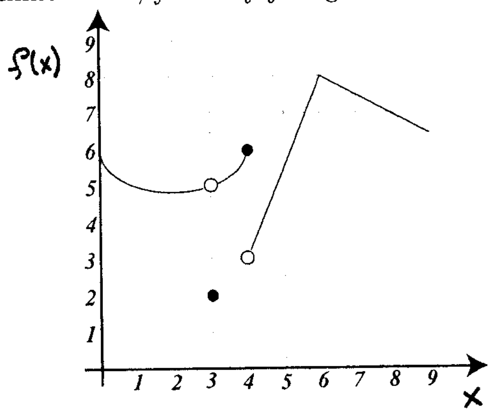
2 iv) $\lim_{y\to 0} \frac{1}{y^4}$ $= \infty$.



$\lim_{y\to 0^+} \frac{1}{y^4} = \infty = \lim_{y\to 0^-} \frac{1}{y^4}$.

(continuation of problem 6)

(b) For the function $f(x)$ graphed below, find the following limits or explain why the limit does not exist. (If a limit exists, you may just give a number without explanation.)



1 point apiece

a) $\lim_{x\to 3} f(x)$ $= 5$

b) $\lim_{x\to 4^-} f(x)$ $= 6$

c) $\lim_{x\to 4^+} f(x)$ $= 3$

d) $\lim_{x\to 4} f(x)$ DNE since $\lim_{x\to 4^+} f(x) = 3 \neq 6 = \lim_{x\to 4^-} f(x)$

e) $\lim_{x\to 6} f(x)$ $= 8$

7. (12 points) If you use a calculator, please make sure that it is in radians.

(a) Accurately graph the function

$$M(t) = \begin{cases} \cos\left(\frac{\pi}{2}t\right), & t < 4 \\ 0.5t - 1, & t \geq 4 \end{cases}$$

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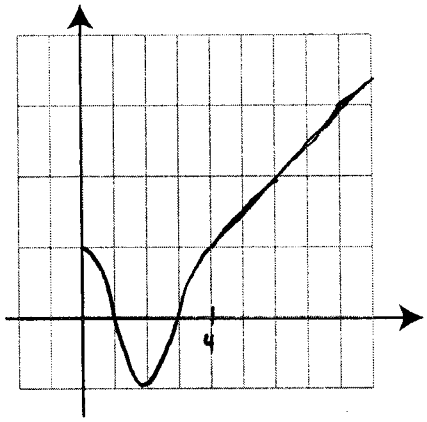

(b) Does $\lim_{t\to 4} M(t)$ exist? If so, find the value of the limit. If not, explain why not.

4 $\lim_{t\to 4^-} M(t) = \lim_{t\to 4^-} \cos\left(\frac{\pi}{2}t\right) = \cos(2\pi) = 1$.

$\lim_{t\to 4^+} M(t) = \lim_{t\to 4^+} (0.5t-1) = 0.5(4) - 1 = 1$.

$\therefore \boxed{\lim_{t\to 4} M(t) = 1}$

(c) Is $M(t)$ continuous at $t = 4$? If so, use the definition of continuity to explain how you know this. If not, explain why it is not continuous there.

Yes, since

4 (i) $M(4)$ is defined $(M(4) = 1)$.

(ii) $\lim_{t\to 4} M(t)$ exists $\left(\lim_{t\to 4} M(t) = 1\right)$

& (iii) $\lim_{t\to 4} (M(t)) = M(4)$.

8. (14 points) Tom Brady forgot to shower after a football game, so foot fungus began to grow. Suppose that the fungus population $f(t)$ on Tom's toes at time t is given by $f(t) = 4t^2 - 4t + 2$.

(a) Find a formula for the slope of the secant line that passes through $(2, f(2))$ and $(2 + \Delta t, f(2 + \Delta t))$.

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$$\text{slope} = \frac{f(2+\Delta t) - f(2)}{(2+\Delta t) - 2} = \frac{[4(2+\Delta t)^2 - 4(2+\Delta t) + 2] - [4(2)^2 - 4(2) + 2]}{\Delta t}$$

$$= \cdots = \frac{12\Delta t + 4\Delta t^2}{\Delta t} = 12 + 4\Delta t.$$

(b) Find the average rate of change in the fungus population between times $t = 2$ and $t = 2.1$.

3

Average Rate of Change = Slope of Secant Line.

Using the result from part a, with $\Delta t = (2.1 - 2) = 0.1$,

average rate of change between $t = 2$ and $t = 2.1$

$= 12 + 4(0.1) = \underline{12.4}$

(c) Use the *definition* of the derivative to find $f'(2)$.

3

$$f'(2) = \lim_{\Delta t \to 0} \frac{f(2+\Delta t) - f(2)}{(2+\Delta t) - 2} \overset{\text{part a}}{=} \lim_{\Delta t \to 0} (12 + 4\Delta t) = \underline{12.}$$

(d) What is the equation of the tangent line to $f(t)$ at time $t = 2$?

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The tangent line is given by

$y - f(2) = f'(2)(x - 2)$;

$y - 10 = 12(x - 2)$;

$y = 12x - 24 + 10$; $\boxed{y = 12x - 14}$