

**Math 155. Homework 5. Sections 2.5, 2.6**

**Hold the Mayo!**

**Arteriosclerosis and the Fourth-Power Relationship in the Hagen-Poiseuille Equation**

After a worksheet by Todd Cooke, Professor of Biology, University of Maryland

**Biological Motivation:** From the website of the National Heart, Lung, and Blood Institute at the National Institute for Health: [http://www.nhlbi.nih.gov/health/dci/Diseases/Cad/CAD\\_All.html](http://www.nhlbi.nih.gov/health/dci/Diseases/Cad/CAD_All.html): “Coronary artery disease (CAD) is a condition in which plaque builds up inside the coronary arteries that supply heart muscle with oxygen-rich blood. Plaque is made up of fat, cholesterol, calcium, and other substances found in the blood. When plaque builds up in the arteries, the condition is called atherosclerosis. Plaque narrows the arteries and reduces blood flow to your heart muscle. It also makes it more likely that blood clots will form and partially or completely block blood flow. When coronary arteries are narrowed or blocked, oxygen-rich blood can’t reach the heart muscle. This can cause angina or a heart attack. Angina is chest pain or discomfort that occurs when not enough blood flows to an area of heart muscle. A heart attack occurs when blood flow to an area of heart muscle is completely blocked. This prevents oxygen-rich blood from reaching that area of heart muscle and causes it to die. Without quick treatment, a heart attack can lead to serious problems and even death. Over time, CAD can weaken heart muscle and lead to heart failure and arrhythmias. Heart failure is a condition in which your heart can’t pump enough blood throughout your body. Arrhythmias are problems with the speed or rhythm of your heartbeat. CAD is the most common type of heart disease. It’s the leading cause of death in the United States for both men and women. Lifestyle changes, medicines, and/or medical procedures can effectively prevent or treat CAD in most people.”

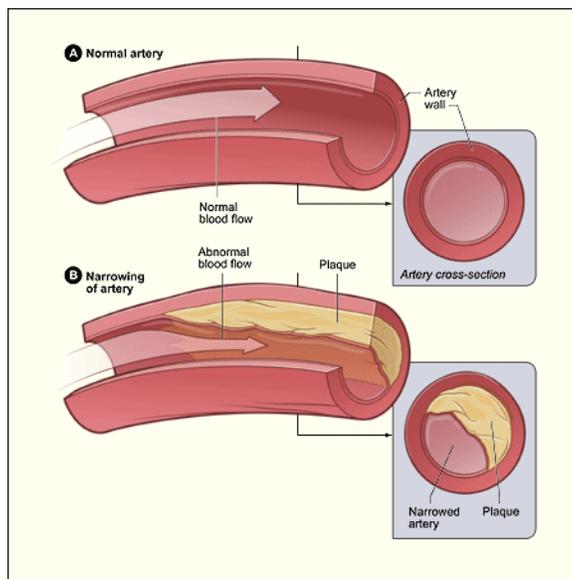


Figure 1: From the website of the National Heart, Lung, and Blood Institute; Figure A shows a normal artery with normal blood flow. Figure B shows an artery with plaque buildup.

**Mathematical Motivation:** *Consequences of Nonlinearity.* The relationship between the flow  $F$  (say, in cubic centimeters per second) of a liquid through a cylindrical tube of radius  $r$  and length  $L$  is given by the Hagen-Poiseuille equation

$$F = \frac{\Delta P}{\eta L} r^4, \quad (1)$$

where  $\eta$  is the viscosity of the fluid and  $\Delta P$  is the pressure difference in the tube. What are the consequences of this *nonlinear* relationship between flow and radius?

**Putting it Together:** Consider the flow of blood with viscosity  $\eta$  through an artery, which we model as a cylindrical tube of radius  $r$ , length  $L$  and a pressure difference  $\Delta P$  produced by the beating heart. Writing  $k = \frac{\Delta P}{\eta L}$ , the Hagen-Poiseuille equation becomes

$$F(r) = kr^4. \quad (2)$$

1. The radius of a typical open artery is 1.5 mm. In the “mild” case of CAD, plaque may line the walls so that it reduces the radius of the open lumen of this occluded artery to 0.75 mm. Assuming that the other variables affecting blood flow remain the same (that is,  $k$  in equation (2) remains the same), use equation (2) to determine the ratio of blood flow in the 50%-occluded artery vs. the open artery. Show your calculations.

2. **Death by Nonlinearity.** Our body attempts to compensate with reduced blood flow (that is, volume/time) in part by increasing the blood pressure. Normal blood pressure varies between 120 mm Hg maximum (systolic) just after pumping of the left ventricle and 80 mm Hg just before the left ventricle pumps again. If your systolic pressure gets above 140 mm Hg, you will be diagnosed with incipient high blood pressure. Higher pressures can mean significant damage to the body’s organs.

How much would the systolic blood pressure have to increase in order to restore the flow in the 50% occluded artery so that it matches the flow in an open artery with a systolic pressure of 120 mm Hg? Show your calculations using equation (1). (Note: the flow in an open artery at the systolic pressure of 120 mm Hg is found using  $\Delta P = 120$  and  $r = 1.5$  in equation (1); you need to find the value of  $\Delta P$  to equal the flow in an open artery if  $r = 0.75$ . That is, you need to solve

$$\frac{120}{\eta L} (1.5)^4 = \frac{\Delta P}{\eta L} (0.75)^4$$

for  $\Delta P$ . )

**3.** An interpretation of your calculations in problems 1 and 2 is that, due to the fourth-power relationship between flow and radius in the Hagen-Poiseuille equation, even small changes in vessel radius have profound effects on the flow. Use differentiation of equation (1) to find formulas for the change in flow with respect to the variables i) radius  $r$ , ii) pressure difference  $\Delta P$ , and iii) viscosity  $\eta$ .

**4. So few for so much!** A large tree adds a new growth ring of xylem cells for conducting water every year. The so-called diffuse-porous woods, such as maples and magnolias, develop many xylem vessels of intermediate radius, or  $45 \mu\text{m}$ . On the other hand, oaks, elms, and other ring-porous woods develop fewer xylem vessels of greater radius, or  $90 \mu\text{m}$ . Let us assume that all features of the conducting vessels are identical in these trees, except for the different radii of their vessels.

(a) In a new growth ring, a maple tree can develop 10,000 new vessels with a radius of  $45 \mu\text{m}$ . How many vessels of radius  $90 \mu\text{m}$  would have to develop in a new growth ring of an oak tree to have an equal rate of total xylem flow as in the maple? Show your calculations using equation (2).

(b) Clearly, it is advantageous to have such large vessels for the purpose of maximal water conduction, but can you think of a plausible reason why the tallest and longest-living trees have many narrow water-conducting cells rather than fewer wide ones?