On Controlling Chaos in an Inflation–Unemployment Dynamical System

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Abstract—Three methods for chaos control are briefly reviewed. Only one of them seems to be applicable to Soliman’s model for unemployment–inflation. © 1999 Elsevier Science Ltd. All rights reserved.

1. BASICS OF CHAOS CONTROL

Chaos [1] is an intrinsic phenomena in many nonlinear systems. Within the last decade, several attempts have been made to control it [2–5]. We begin by briefly reviewing three methods for chaos control. It will then be shown that only one of them seems to work for Soliman’s model for unemployment–inflation [6].

Ott et al.’s method (OGY) [2] assumes that the system is described by

$$\tilde{r}_{t+1} = \tilde{f}(\tilde{r}_t, \tilde{p})$$

where \(\tilde{p}\) are the parameters of the system. If the system is chaotic, then the OGY method states that they should be varied by \(\delta\tilde{p}\), such that the total variation of the function \(\tilde{f}(\tilde{r}_t, \tilde{p} + \delta\tilde{p})\) is along the stable manifold of the system. Hence, its projection on the unstable manifold is zero. This gives the relation

$$\delta p_j(t) = -\lambda(e^\cdot \delta r_j)/(e^\cdot \delta e_j)$$

where \(\delta p_j(t)\) is the variation of the \(j\)th component of the parameters at time \(t\), \(\delta r_j = \tilde{r} - \bar{r}\), \(\bar{r}\) is the steady state about which one wishes to stabilize the system. \(\lambda\) is the positive eigenvalue of the Jacobian matrix \(A\),

$$A = [\tilde{c}_f, \tilde{c}_r]$$

and \(e\) is the contravariant eigenvector of \(A\) corresponding to \(\lambda\), i.e., if \(Ae_j = \lambda_j e_j\), \(j = 1\) & \(j \neq \lambda\).

In the Guemes–Matias method (GM) [3, 5], the states \(\tilde{r}_j\) are multiplied by constant factors, \(K = \text{diag}(k_1, k_2, \ldots, k_n)\) every \(L\) time steps. Hence, the controlled system is given by either one of the equations

$$\tilde{r}_{t+1} = Kf(\tilde{r}_t, \tilde{p})$$
$$\tilde{r}_{t+1} = \tilde{f}(\tilde{r}_t, \tilde{p})$$
where the first equation is valid if \( t \) is an integer multiple of \( L \), else the second equation is used. The factor, \( K \), and the interval, \( L \), are chosen such that the new system (4) is stable even though the original system (1) is chaotic.

The previous two methods and many others attempt to control the system within the chaotic region. Recently, a method has been proposed \([4]\) to guide the system gradually to the stability zone. It has the advantage that, once the system reaches the stability region, control can be turned off. The idea is to determine the stability zone then, if the chaotic region exists at larger values of the parameters, choose the parameters’ variation to be

\[
\Delta \tilde{p}_t = -|\delta \tilde{p}_t|, \quad (5.1)
\]

else

\[
\Delta \tilde{p}_t = |\delta \tilde{p}_t|, \quad (5.2)
\]

where \( \delta \tilde{p}_t \) is given in (2). if the variation is larger (in absolute value) than an allowed value (say \( \bar{x} \)), then set

\[
\Delta \tilde{p}_t = \bar{x}. \quad (5.3)
\]

### 2. AN UNEMPLOYMENT–INFLATION MODEL

The following model will be studied \([6]\):

\[
\begin{align*}
u_{t+1} &= u_t - b(m - \pi_t), \\
\pi_{t+1} &= \pi_t - (1 - c)f(u_t) + f(u_t - b(m - \pi_t)),
\end{align*}
\]

\[
f(u) = \beta_1 + \beta_2 e^{-u},
\]

where \( u, \pi, b, c > 0 \).

It can be shown by rescaling that the parameter, \( m \), does not affect the stability analysis. Following Soliman’s choice, we set

\[
m = 2, \beta_1 = -2.5, \beta_2 = 20. \quad (7)
\]

The system (6) has a unique steady state

\[
\pi = 2, u = \ln\left(\frac{-\beta_1}{\beta_2}\right).
\]

The Jacobian matrix, \( \mathbf{J} \), of the linearized system of (8) is

\[
\mathbf{J} = \begin{bmatrix} 1 & b \\ c\beta_1 & 1 + b\beta_1 \end{bmatrix}.
\]

The eigenvalues of the Jacobian matrix are the solution of the characteristic equation,

\[
\lambda^2 - b\lambda + \gamma = 0,
\]

where \( \beta = 2 - 5b/2 \) and \( \gamma = 1 - 5b/2 + 5cb/2 \).

From Jury stability conditions, the absolute values of the eigenvalues are less than unity in absolute if and only if
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Fig. 1. Bifurcation diagram of the state \( u \) at \( c = 0.18 \) and \( 0.8 < b < 1.24 \).

These conditions give the stability region of the fixed point. This region of stability of the equilibrium is bounded by the lines \( c = 0 \), \( c = 1 \), \( b = 0 \) and the curve \( (b(2-c)) < 1.6 \).

The eigenvalues of the Jacobian at the given values of \( \beta_1 \) and \( \beta_2 \) are

\[
\begin{align*}
\lambda_1 &= 1 - \frac{5b}{2} + \frac{1}{4} \sqrt{25b^2 - 40bc} \\
\lambda_2 &= 1 - \frac{5b}{2} - \frac{1}{4} \sqrt{25b^2 - 40bc}.
\end{align*}
\]

Lyapunov exponents are used to detect chaos regions; for example, if we take \( c = 0.18 \), we find chaotic behaviour as shown in Fig. 1.

Here, we will concentrate on the values \( b = 1.2 \) and \( c = 0.18 \) for chaos control.

3. CONTROLLING CHAOS IN THE MODEL

Applying the OGY method, the parameters \( b \) and \( c \) should be varied by

\[
\begin{align*}
\delta b_i &= -\lambda_i(\delta u_i, \delta \pi_i) / \left( \frac{\partial g}{\partial b} \delta b + \frac{\partial h}{\partial b} \delta h \right) \\
\delta c_i &= -\lambda_i(\delta u_i, \delta \pi_i) / \left( \frac{\partial g}{\partial c} \delta c + \frac{\partial h}{\partial c} \delta c \right),
\end{align*}
\]

where

\[
\begin{align*}
g(u, \pi) &= u - b(2 - \pi) \\
h(u, \pi) &= \pi - (1 - c)f(u) + f(u - (2 - \pi)).
\end{align*}
\]

It is easy to see that at the steady state (8), one has
\[ \ddot{g} = \frac{\partial h}{\partial b} = \frac{\partial h}{\partial c} = 0. \]  

Hence, OGY seems inapplicable to this model.

Applying the GM method, and for simplicity letting \( L = 1 \), then the new dynamical equations are

\[ u_{t+1} = k_1 g(u, \pi), \]
\[ \pi_{t+1} = k_2 h(u, \pi). \]

The stability constraints become

\[ | - k_1 + k_2 (2.5b - 1) | < 1 \]
\[ k_1 k_2 [1 - 2.5b (1 - c)] + 1 > | - k_1 + k_2 (2.5b - 1) |. \]

or simplicity, setting \( k_1 = k_2 = k \) and concentrating on the values \( k \approx 0.5 \), one finds that the system is stable for \( k \approx 0.5 \). Practically, it is extremely difficult to reduce both inflation and unemployment by this factor for such a short step \( (L = 1) \).

Moreover, even for this value of \( k \), chaos is not controlled. The reason is that this reduction guides the system to the state \( u = \pi = 0 \), which is not a steady state of the system \( (6) \). Hence, the GM method seems inapplicable to this model.

The third method \( (3) \) depends on observing that chaos exists because the values of \( b \) and \( c \) has exceeded the stable zone. The formula \( (4.2) \) will be used since both \( (5.1) \) and \( (5.2) \) have the same problems as OGY. The reduction

\[ b_{t+1} = 0.999b_t, \quad c_{t+1} = 0.999c_t, \]

is applied whenever \( t \) is an integer multiple of \( L \) (taken here to be \( L = 3 \) or \( L = 5 \)). This procedure guides the system to the steady state \( (8) \). Once this is reached, the control \( (16) \) is turned off. Thus, chaos control is successful.

The methods presented here can be applied to many chaotic dynamical systems.

REFERENCES