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The Weierstrass-Enneper Representations

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Abstract. The Weierstrass-Enneper Representations are a great link between several branches of mathematics. They provide a way to study surfaces using both geometry and complex analysis. The Weierstrass-Enneper Representation for minimal surfaces says that any minimal surface may be represented by complex holomorphic functions. With the use of differential forms, this idea may be generalized to constant mean curvature surfaces, which yields Hamiltonian systems. The ability to study a problem from several different angles can be a very useful tool in mathematics, which makes the Weierstrass-Enneper Representations an exciting discovery.

Keywords: Weierstrass-Enneper Representations, minimal surfaces, constant mean curvature surfaces, holomorphic functions, differential forms, Hamiltonian systems

1 Introduction

Mathematics is divided into many different categories and subjects. Sometimes it seems as if these subjects do not connect, especially when studied in high school or college classes that do not feel as if they overlap. It is always exciting, then, when a connection between two seemingly unrelated areas of math is made. In reality, the many different "subjects" of math are intertwined in beautiful ways. Sometimes these connections are obvious and at other times they take some thought and imagination to discover. Karl Weierstrass and Alfred Enneper discovered one of these connections. Their discovery is now known as the Weierstrass-Enneper Representations. They link together complex analysis, differential geometry, and Hamiltonian systems. Weierstrass and Enneper figured out that minimal surfaces can be represented by holomorphic and meromorphic complex functions. This idea can then be generalized to constant mean curvature surfaces, and this generalization produces Hamiltonian systems. In order to dive into all of this, some background information about minimal surfaces, complex functions, and Hamiltonian systems is needed.

2 Minimal Surfaces

A surface M is called a minimal surface if the mean curvature, usually called H, is zero. The mean curvature is the average of the principal curvatures of that surface, so if we call the principal curvatures k_1 and k_2, then

$$H = \frac{k_1 + k_2}{2}. \tag{1}$$

The principal curvatures at a point p on a surface are the curvatures in the directions of maximal and minimal curvature at that point. They can be computed as the eigenvalues of the shape operator (S_p), which is a linear transformation from the tangent space of the surface at that point to itself ($S_p : T_pM \to T_pM$) [1].

Let a surface $M \subseteq \mathbb{R}^3$ be parameterized by $\vec{x}(u,v) : \Omega \subseteq \mathbb{R}^2 \to M$. Then the unit normal vector to the surface is $\vec{N} = \frac{\vec{x_u} \times \vec{x_v}}{|\vec{x_u} \times \vec{x_v}|}$. The following equations can be used to compute the mean curvature and provide a way to define the shape operator as a matrix [1, 2]. Define E, F, G, l, m, and n as

$$\begin{aligned}
E &= \vec{x_u} \cdot \vec{x_u}, \\
F &= \vec{x_u} \cdot \vec{x_v}, \\
G &= \vec{x_v} \cdot \vec{x_v}, \\
l &= \vec{x_{uu}} \cdot \vec{N}, \\
m &= \vec{x_{uv}} \cdot \vec{N}, \\
n &= \vec{x_{vv}} \cdot \vec{N}.
\end{aligned} \tag{2}$$

Then the shape operator is

$$S = \frac{1}{EG - F^2} \begin{pmatrix} Gl - Fm & Gm - Fn \\ Em - Fl & En - Fm \end{pmatrix}, \tag{3}$$

and the mean curvature is

$$H = \frac{En + Gl - 2Fm}{2(EG - F^2)} = \frac{1}{2}\mathrm{tr}(S). \tag{4}$$

These equations work for any patch $\vec{x}(u,v)$, but sometimes it is useful to use a patch with special properties.

3 Isothermal Patch

If $\vec{x}(u,v) : \Omega \to M$ is a patch such that $E = G$ and $F = 0$, it is called an isothermal patch. Geometrically this means that $\vec{x_u}$ and $\vec{x_v}$ are orthogonal, so angles are preserved, and $\vec{x}$ stretches the patch the same amount in the u and v directions. If $\vec{x}$ is an isothermal patch, then

$$H = \frac{En + El}{2E^2} = \frac{n + l}{2E}, \tag{5}$$

so the mean curvature is very easy to compute. There is a theorem that states that isothermal coordinates exist on any minimal surface $M \subseteq \mathbb{R}^3$. In fact, any surface can be parameterized using an isothermal patch. The Weierstrass-Enneper Representation for minimal surfaces requires that the minimal surfaces be represented by isothermal patches [1].

4 Harmonic Functions

Another type of patch that plays a role in the Weierstrass-Enneper Representations is a harmonic patch. A real function $x(u,v)$ is harmonic if its second-order partial derivatives are continuous and $\triangle x := \frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2} = 0$. A theorem in differential geometry states that if $\vec{x}(u,v)$ is isothermal, then $\triangle \vec{x} = (2EH)\vec{N}$. The following corollary to this theorem is important in the proof of the

Weierstrass-Enneper Representation for minimal surfaces [1].

Corollary: A surface $M : \vec{x}(u,v) = (x^1(u,v), x^2(u,v), x^3(u,v))$, with isothermal coordinates is minimal if and only if x^1, x^2, and x^3 are all harmonic.

Proof [1]: $(\Longrightarrow)$ If M is minimal, then $H = 0 \Longrightarrow \triangle\vec{x} = (2EH)\vec{N} = 0 \Longrightarrow x^1, x^2, x^3$ are harmonic.

$(\Longleftarrow)$ If x^1, x^2, x^3 are harmonic, then $\triangle\vec{x} = 0 \Longrightarrow (2EH)\vec{N} = 0$.
Now $\vec{N}$ is the unit normal vector, so $\vec{N} \neq 0$ and $E = \vec{x_u} \cdot \vec{x_v} = |\vec{x_u}|^2 \neq 0$.
So $H = 0 \Longrightarrow$ M is minimal.

QED

Intuitively this corollary makes sense because the curvature (κ) of a curve in a plane is essentially the rate that the tangent vector to the curve is changing. For a curve (α) parameterized by arc-length, $\kappa = |\frac{dT}{ds}| = |\frac{d^2\alpha}{ds^2}|$. Since $\vec{x}(u,v)$ is not parameterized by arc-length, the principle curvatures are not exactly the magnitude of the second derivatives $|\vec{x}_{uu}|$ and $|\vec{x}_{vv}|$, but they are certainly related. So it makes sense to say that if $x^j_{uu} + x^j_{vv} = 0$ for $j \in \{1,2,3\}$, then $k_1 + k_2 = 0$ and vice versa.

5 Holomorphic and Meromorphic Complex Functions

The Weierstrass-Enneper Representations do not only depend on differential geometry concepts, it is also important to understand some complex analysis. A complex function $f(z)$ is holomorphic at a point z_0 if $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists, so f is holomorphic in a region if it is differentiable at every point in that region. A complex function $g(z)$ is meromorphic in a region if it is holomorphic everywhere in that region except at isolated singularities and all of these singularities are poles. A point z_0 is a pole if $f \to \infty$ as $z \to z_0$. A complex number may be expressed as $z = u + iv$ where $u, v \in \mathbb{R}$, and its complex conjugate is $\overline{z} = u - iv$. The derivatives of z and $\overline{z}$ may be expressed as $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$ and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$. If $f : U \to V$ is holomorphic and bijective, it is called a conformal map. Conformal maps preserve angles [3]. Isothermal patches also preserve angles, so this is the first connection between differential geometry and complex analysis that has been mentioned so far [4]. If a minimal surface can be represented by an isothermal patch, could it also be represented by a holomorphic function?

6 The Weierstrass-Enneper Representation for Minimal Surfaces

With all of this background information about minimal surfaces, isothermal patches, harmonic functions, and holomorphic and meromorphic functions, the Weierstrass-Enneper Representation for minimal surfaces may be constructed. First, it would be good to look at an example. Since it is called the Weierstrass-Enneper Representation, Enneper's Surface makes a great example [4].

Enneper's Surface

The most common parameterization for Enneper's surface is

$$\vec{x}(u,v) = (u - \frac{1}{3}u^3 + uv^2, -v - u^2v + \frac{1}{3}v^3, u^2 - v^2). \tag{6}$$

First show that this is an isothermal patch.

$$\begin{aligned}\vec{x_u} &= (1-u^2+v^2, -2uv, 2u)\\ \vec{x_v} &= (2uv, -1-u^2+v^2, -2v)\end{aligned}$$

$$\begin{aligned}E &= \vec{x_u}\cdot\vec{x_u} = 1+2u^2+2v^2+2u^2v^2+u^4+v^4\\ G &= \vec{x_v}\cdot\vec{x_v} = 1+2u^2+2v^2+2u^2v^2+u^4+v^4\\ F &= \vec{x_u}\cdot\vec{x_v} = 2uv(1-u^2+v^2)-2uv(-1-u^2+v^2)-4uv = 0\end{aligned}$$

Since $E = G$ and $F = 0$, $\vec{x}(u,v)$ is isothermal.

Now let $z = u + iv$ and $\vec{\phi} = \vec{x_u} - i\vec{x_v}$.

Then

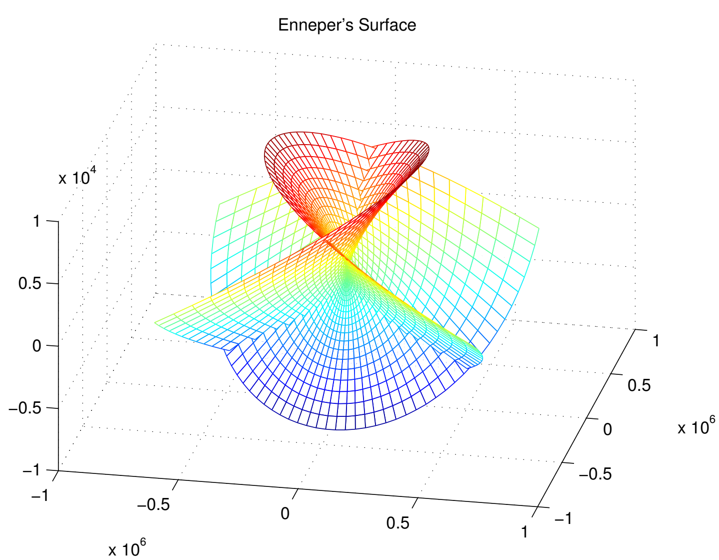
$$\begin{aligned}\vec{\phi} &= (1-u^2+v^2-i2uv, -2uv-i(-1-u^2+v^2), 2u+i2v)\\ &= (1-(u+iv)^2, i(1+(u+iv)^2), 2(u+iv))\\ &= (1-z^2, i(1+z^2), 2z).\end{aligned}$$

Notice that $\phi^1(z) = 1-z^2$, $\phi^2(z) = i(1+z^2)$, and $\phi^3(z) = 2z$ are all holomorphic, so Enneper's surface can be represented by holomorphic functions [4].
Is it possible to go backwards? Given $\vec{\phi}$, can Enneper's surface be derived? Weierstrass figured out that, yes it is possible to obtain Enneper's surface from $\vec{\phi}$.

We know $\vec{\phi} = \vec{x_u} - ix_v$ and $\phi^1(z) = 1-z^2$, $\phi^2(z) = i(1+z^2)$, and $\phi^3(z) = 2z$, and we want $\vec{x}(u,v)$ to be real-valued.

Let
$$\begin{aligned}x^1 &= Re(\textstyle\int(1-z^2)dz)\\ &= Re(z-\tfrac{1}{3}z^3)\\ &= Re(u+iv-\tfrac{1}{3}(u+iv)^3)\\ &= u-\tfrac{1}{3}u^3+uv^2\end{aligned}$$
$$\begin{aligned}x^2 &= Re(\textstyle\int i(1+z^2)dz)\\ &= Re(i(z+\tfrac{1}{3}z^3))\\ &= Re(i(u+iv+\tfrac{1}{3}(u+iv)^3))\\ &= -v-u^2v+\tfrac{1}{3}v^3\end{aligned}$$
$$\begin{aligned}x^3 &= Re(\textstyle\int 2zdz)\\ &= Re(z^2)\\ &= Re((u+iv)^2)\\ &= u^2-v^2.\end{aligned}$$

Then we get $\vec{x}(u,v) = (u-\frac{1}{3}u^3+uv^2, -v-u^2v+\frac{1}{3}v^3, u^2-v^2)$, which is Enneper's surface [4]!



Enneper's surface provides a great example of how a minimal surface may be represented by holomorphic functions, but now this idea needs to be generalized to all minimal surfaces.

From Isothermal Patches to Holomorphic Functions

Let M be a minimal surface described by an isothermal patch $\vec{x}(u,v)$.
Let $z = u + iv$, so then $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$, $\overline{z} = u - iv$, and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$.
Notice that $z + \overline{z} = 2u$ and $z - \overline{z} = i2v$, so

$$u = \frac{z+\overline{z}}{2} \qquad (7)$$
$$v = \frac{z-\overline{z}}{2i}.$$

This means that $\vec{x}(u,v)$ may be written as

$$\vec{x}(z,\overline{z}) = (x^1(z,\overline{z}), x^2(z,\overline{z}), x^3(z,\overline{z})), \qquad (8)$$

and the derivative of the j^{th} component is

$$\frac{\partial x^j}{\partial z} = \frac{1}{2}(x_u^j - ix_v^j). \qquad (9)$$

Define

$$\vec{\phi} = \frac{\partial \vec{x}}{\partial z} = (x_z^1, x_z^2, x_z^3) \qquad (10)$$
$$(\phi)^2 = (x_z^1)^2 + (x_z^2)^2 + (x_z^3)^2. \qquad (11)$$

Then $(\phi^j)^2 = (x_z^j)^2 = (\frac{1}{2}(x_u^j - ix_v^j))^2 = \frac{1}{4}((x_u^j)^2 - (x_v^j)^2 - 2ix_u^j x_v^j)$,
so $(\phi)^2 = \frac{1}{4}(\sum_{j=1}^{3}((x_u^j)^2 - (x_v^j)^2 - 2ix_u^j x_v^j))$
$= \frac{1}{4}(|\vec{x}_u|^2 - |\vec{x}_v|^2 - 2i\vec{x}_u \cdot \vec{x}_v)$
$= \frac{1}{4}(E - G - 2iF)$.

Since $\vec{x}$ is isothermal, $(\phi)^2 = \frac{1}{4}(E - E) = 0$ [1].

The following theorem relates minimal surfaces to holomorphic functions.

Theorem: Suppose M is a surface with patch $\vec{x}$. Let $\vec{\phi} = \frac{\partial \vec{x}}{\partial z}$ and suppose $(\phi)^2 = 0$ (i.e., $\vec{x}$ is isothermal). Then M is minimal if and only if each ϕ^j is holomorphic [1].

The proof of this theorem requires the following lemma.

Lemma: $\frac{\partial}{\partial \overline{z}}(\frac{\partial \vec{x}}{\partial z}) = \frac{1}{4} \triangle \vec{x}$ [1, 3].

Proof of lemma: $\frac{\partial}{\partial \overline{z}}(\frac{\partial \vec{x}}{\partial z}) = \frac{\partial}{\partial \overline{z}}(\frac{1}{2}(\frac{\partial \vec{x}}{\partial u} - i\frac{\partial \vec{x}}{\partial v}))$
$= \frac{1}{2}(\frac{1}{2}(\frac{\partial}{\partial u}(\frac{\partial \vec{x}}{\partial u} - i\frac{\partial \vec{x}}{\partial v}) + i\frac{\partial}{\partial v}(\frac{\partial \vec{x}}{\partial u} - i\frac{\partial \vec{x}}{\partial v})))$
$= \frac{1}{4}(\frac{\partial^2 \vec{x}}{\partial u^2} - i\frac{\partial \vec{x}}{\partial u}\frac{\partial \vec{x}}{\partial v} + i\frac{\partial \vec{x}}{\partial u}\frac{\partial \vec{x}}{\partial v} + \frac{\partial^2 \vec{x}}{\partial v^2})$
$= \frac{1}{4}(\frac{\partial^2 \vec{x}}{\partial u^2} + \frac{\partial^2 \vec{x}}{\partial v^2})$
$= \frac{1}{4} \triangle \vec{x}$.

QED

Another useful tool in proving the above theorem is a theorem from complex analysis that says f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ [3].

Proof of the above theorem: ($\Longrightarrow$) If M is minimal, then, by the corollary proven in the harmonic function section, x^j is harmonic for $j \in \{1,2,3\}$.

$$\begin{aligned} x^j \text{ harmonic} &\Longrightarrow \triangle\vec{x} = 0 \\ &\Longrightarrow \tfrac{1}{4}\triangle\vec{x} = 0 \\ &\Longrightarrow \tfrac{\partial}{\partial\bar{z}}\left(\tfrac{\partial\vec{x}}{\partial z}\right) = 0 \text{ by the above lemma} \\ &\Longrightarrow \tfrac{\partial\vec{\phi}}{\partial\bar{z}} = 0. \end{aligned}$$

By the theorem from complex analysis, since $\frac{\partial}{\partial\bar{z}}(\frac{\partial\vec{x}}{\partial z}) = 0$, ϕ^j is holomorphic.

($\Longleftarrow$) If ϕ^j is holomorphic, then $\frac{\partial\vec{\phi}}{\partial\bar{z}} = 0 \Longrightarrow \frac{\partial}{\partial\bar{z}}(\frac{\partial\vec{x}}{\partial z}) = \frac{1}{4}\triangle\vec{x} = 0$

$$\begin{aligned} &\Longrightarrow \triangle\vec{x} = 0 \\ &\Longrightarrow x^j \text{ is harmonic.} \end{aligned}$$

x^j harmonic $\Longrightarrow$ M is minimal.

QED

Now any minimal surface may be represented using $\vec{\phi}$ with holomorphic components and $(\phi)^2 = 0$. Given $\vec{\phi}$, how is an isothermal patch $\vec{x}$ for M constructed? The following corollary to the theorem proven above shows that the components of $\vec{\phi}$ may be integrated to obtain the components of $\vec{x}$ [1].

Corollary: $x^j(z,\bar{z}) = c_j + 2Re(\int \phi^j dz)$ [1].

Proof of corollary [1]: $z = u + iv \Longrightarrow dz = du + idv$

$$\phi^j dz = \tfrac{1}{2}(x_u^j - ix_v^j)(du + idv) = \tfrac{1}{2}((x_u^j du + x_v^j dv) + i(x_u^j dv - x_v^j du))$$

$$\overline{\phi}^j d\bar{z} = \tfrac{1}{2}(x_u^j + ix_v^j)(du - idv) = \tfrac{1}{2}((x_u^j du + x_v^j dv) - i(x_u^j dv - x_v^j du))$$

Then we have
$$\begin{aligned} dx^j &= \tfrac{\partial x^j}{\partial z}dz + \tfrac{\partial x^j}{\partial\bar{z}}d\bar{z} \\ &= \phi^j dz + \overline{\phi}^j d\bar{z} \\ &= \tfrac{1}{2}(x_u^j du + x_v^j dv) + \tfrac{1}{2}(x_u^j du + x_v^j dv) \\ &= x_u^j du + x_v^j dv \\ &= 2Re(\phi^j dz) \end{aligned}$$

$$\Longrightarrow x^j = 2Re(\textstyle\int \phi^j dz) + c_j.$$

QED

Now we know how to construct $\vec{x}$ if we have $\vec{\phi}$, but what is $\vec{\phi}$ for a general minimal surface? We need each component, ϕ^j to be holomorphic and $(\phi)^2 = 0$. A nice way to construct $\vec{\phi}$ is as follows [1].

Let f be a holomorphic function and g be a meromorphic function such that fg^2 is holomorphic. Let

$$\begin{aligned} \phi^1 &= \frac{1}{2}f(1 - g^2) \\ \phi^2 &= \frac{i}{2}f(1 + g^2) \\ \phi^3 &= fg. \end{aligned} \tag{12}$$

Then ϕ^1, ϕ^2, and ϕ^3 are holomorphic and $(\phi)^2 = \frac{1}{4}f^2(1-g^2)^2 - \frac{1}{4}f^2(1+g^2)^2 + f^2g^2 = 0$ [1].

Now we have everything we need to understand the Weierstrass-Enneper Representation for minimal surfaces.

The Weierstrass-Enneper Representation for Minimal Surfaces

Theorem: The Weierstrass-Enneper Representation [1]: If f is holomorphic on a domain D, g is meromorphic on D, and fg^2 is holomorphic on D, then a minimal surface is defined by $\vec{x}(z,\overline{z}) = (x^1(z,\overline{z}), x^2(z,\overline{z}), x^3(z,\overline{z}))$, where

$$\begin{aligned} x^1(z,\overline{z}) &= Re(\int f(1-g^2)dz) \\ x^2(z,\overline{z}) &= Re(\int if(1+g^2)dz) \\ x^3(z,\overline{z}) &= Re(\int 2fgdz). \end{aligned} \tag{13}$$

In the Enneper's surface example, we need $f = 1$ and $g = z$. Then $\vec{\phi} = (1-z^2, i(1+z^2), 2z) = (f(1-g^2), if(1+g^2), 2fg)$.

There is another way to write Weierstrass-Enneper using just one holomorphic function that is a composition of functions. If g is holomorphic with g^{-1} also holomorphic, then set $\tau = g$ which means $\frac{d\tau}{dz} = \frac{dg}{dz}$, so $d\tau = dg$. Define $F(\tau) = f/\frac{dg}{dz} = f\frac{dz}{dg}$. Then $F(\tau)d\tau = f(\frac{dz}{dg})(dg) = fdz$. Substitute τ for g and $F(\tau)d\tau$ for fdz in the Weierstrass-Enneper Representation to get the following version of Weierstrass-Enneper [1].

Theorem: Weierstrass-Enneper Representation II: For any holomorphic function $F(\tau)$, a minimal surface is defined by $\vec{x}(z,\overline{z}) = (x^1(z,\overline{z}), x^2(z,\overline{z}), x^3(z,\overline{z}))$ where

$$\begin{aligned} x^1(z,\overline{z}) &= Re(\int (1-\tau^2)F(\tau)dz) \\ x^2(z,\overline{z}) &= Re(\int i(1+\tau^2)F(\tau)dz) \\ x^3(z,\overline{z}) &= Re(\int 2\tau F(\tau)dz). \end{aligned} \tag{14}$$

A good example of using this version of Weierstrass-Enneper is the helicoid.

The Helicoid

A helicoid may be obtained from $F(\tau) = \frac{i}{2\tau^2}$ where $\tau = e^z$ [1]. Notice that $\tau = e^z$, $\tau^{-1} = Log(z)$, and $F(e^z) = \frac{i}{2e^{2z}}$ are all holomorphic on the domain of $Log(z)$. I have used $Log(z)$ instead of $log(z)$ because $Log(z)$ is the principal branch of the log and branches of the log are holomorphic, but log itself is not. Now compute $\vec{x}(u,v)$ in the following way.

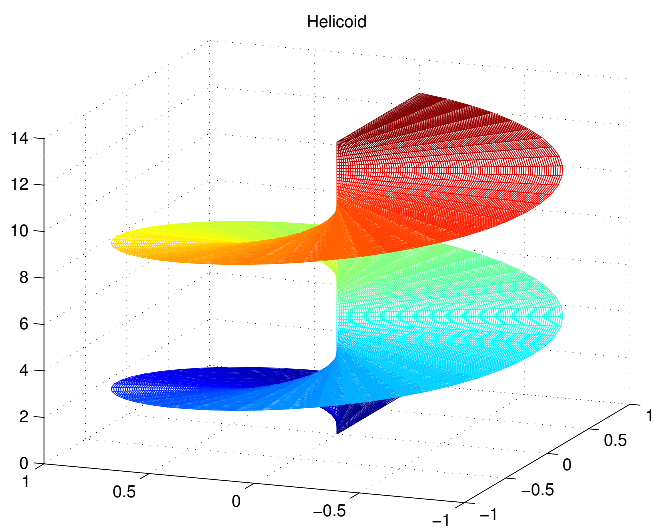
$$\begin{aligned} x^1 &= Re(\int (1-\tau^2)\tfrac{i}{2\tau^2}d\tau) \\ &= Re(\tfrac{-i}{2\tau} - \tfrac{i}{2}\tau) \\ &= Re(\tfrac{-i}{2}(e^{-z}+e^z)) \\ &= Re(\tfrac{-i}{2}(e^{-(u+iv)}+e^{u+iv})) \end{aligned}$$

$$
\begin{aligned}
&= Re(\tfrac{-i}{2}(e^{-u}(cos(-v) + isin(-v)) + e^{u}(cos(v) + isin(v)))) \\
&= Re(\tfrac{-i}{2}e^{-u}cos(-v) + \tfrac{1}{2}e^{-u}sin(-v) - \tfrac{i}{2}e^{u}cos(v) + \tfrac{1}{2}e^{u}sin(v)) \\
&= \tfrac{1}{2}e^{-u}sin(-v) + \tfrac{1}{2}e^{u}sin(v)
\end{aligned}
$$

$$
\begin{aligned}
x^2 &= Re(\textstyle\int i(1+\tau^2)\frac{i}{2\tau^2}d\tau) \\
&= Re(\tfrac{1}{2\tau} - \tfrac{1}{2}\tau) \\
&= Re(\tfrac{1}{2}(e^{-z} - e^{z})) \\
&= Re(\tfrac{1}{2}(e^{-(u+iv)} - e^{u+iv})) \\
&= Re(\tfrac{1}{2}(e^{-u}(cos(-v) + isin(-v)) - e^{u}(cos(v) + isin(v)))) \\
&= Re(\tfrac{1}{2}e^{-u}cos(-v) + \tfrac{i}{2}e^{-u}sin(-v) - \tfrac{1}{2}e^{u}cos(v) + \tfrac{i}{2}e^{u}sin(v)) \\
&= \tfrac{1}{2}e^{-u}cos(-v) - \tfrac{1}{2}e^{u}cos(v)
\end{aligned}
$$

$$
\begin{aligned}
x^3 &= Re(\textstyle\int 2\tau(\frac{i}{2\tau^2})d\tau) \\
&= Re(iLog|\tau|) \\
&= Re(iLog|e^{z}|) \\
&= Re(iz) \\
&= Re(i(u+iv)) \\
&= Re(iu - v) \\
&= -v
\end{aligned}
$$

So $\vec{x}(u,v) = (\frac{1}{2}(e^{-u}sin(-v) + e^{u}sin(v)), \frac{1}{2}(e^{-u}cos(-v) - e^{u}cos(v)), -v)$ is an isothermal patch for the helicoid.



Now anyone who's hobby is finding minimal surfaces can easily find them by integrating holomorphic functions. What about other types of surfaces? The Weierstrass-Enneper Representation may be generalized to constant mean curvature surfaces, which yields some exciting results. Before doing so, a different derivation of the Weierstrass-Enneper Representations integrating the machinery of differential forms is necessary.

7 Weierstrass-Enneper Derivation Using Differential Forms

Consider a surface $\Omega \subset \mathbb{R}^3$, with orthonormal frame $\{e_1, e_2, e_3\}$ such that e_3 is the normal vector at each point on Ω. For 1-forms ω^1 and ω^2 on the original patch there exist $\omega^{1'}$, $\omega^{2'}$ on Ω such that

$$\begin{pmatrix} \omega^{1'} \\ \omega^{2'} \end{pmatrix} = T \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix},$$

where T is the shape operator, as defined in Eq. 3 [5]. If T is given by

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix},$$

then, by Eq. 4, Ω is a minimal surface if and only if $t_{11} + t_{22} = 0$.
Define the 1-form

$$\tau = (e_1 - ie_2)\omega^1 + (e_2 + ie_1)\omega^2.$$

Then,

$$d\tau = i(t_{11} + t_{22})e_3 \;\; \omega^1 \wedge \omega^2.$$

So, the following are equivalent:

1. Ω is a minimal surface.
2. $t_{11} + t_{22} = 0$.
3. $d\tau = 0$
4. τ is a closed form.

Define the function f such that

$$\omega^1 + i\omega^2 = f dz \;\text{ and therefore }\; \omega^1 - i\omega^2 = \overline{f} d\overline{z}. \tag{15}$$

Taking the product of the two equations in Eq. 15 gives

$$\begin{aligned} |f|^2 dz \wedge d\overline{z} &= \omega^1 \wedge \omega^1 - i\omega^1 \wedge \omega^2 + i\omega^2 \wedge \omega^1 + \omega^2 \wedge \omega^2 \\ |f|^2 dz \wedge d\overline{z} &= -2i \;\; \omega^1 \wedge \omega^2 \\ dz \wedge d\overline{z} &= -\frac{2i}{|f|^2} \;\; \omega^1 \wedge \omega^2 \neq 0 \end{aligned}$$

It follows that z can be used as a local coordinate on Ω, which makes Ω into a Riemann surface. Finally, define $F(z, \overline{z}) = (e_1 - ie_2)f$. It follows that

$$F dz = (e_1 - ie_2) f dz = (e_1 - ie_2)(\omega^1 + i\omega^2) = \tau.$$

Therefore, $d\tau = 0$ if and only if $d(F dz) = dF \wedge dz = 0$. This gives the condition that F is a function of z only, or $\frac{\partial F}{\partial \overline{z}} = 0$. It follows that Ω is a minimal surface if and only if F is holomorphic. If Ω is indeed a minimal surface, then there exists a holomorphic function $v(z)$ such that $v'(z) = F(z)$. Using v, one can define the 1-form $\xi = dv$, which gives rise to

$$\Re(\xi) = \Re(dv) = e_1\omega^1 + e_2\omega^2 = dx,$$

for a local position vector x on the surface. It follows that there exists some real-valued vector y such that

$$v(z) = x(z) + iy(z).$$

Conversely, for any holomorphic $\mathbb{C}^3$-valued function v, the surface $\Re(v)$ is a minimal surface. This gives rise to Eq. 13 [5].

8 Generalization to Constant Mean Curvature Surfaces

Let's try to generalize the new derivation for minimal surface to surfaces of constant mean curvature. Let $H = c$, for some constant c. Then $t_{11} + t_{22} = 2H = 2c$. Thus,

$$d\tau = 2ice_3 \ \omega^1 \wedge \omega^2.$$

Since $d\tau = d(Fdz)$,

$$d(Fdz) = dF \wedge dz = 2ice_3 \ \omega^1 \wedge \omega^2 \neq 0.$$

Therefore $\frac{\partial F}{\partial \overline{z}} \neq 0$, so F is not a holomorphic function. **OH NO!** There is no general Weierstrass Enneper representation for constant-mean curvature surfaces. However, this does not mean that there are not other ways of representing constant mean curvature surfaces via holomorphic functions. Indeed, if a system has a Hamiltonian structure, than the corresponding functions can be found.

Example: Start with the linear system

$$\psi_{1z} = p\psi_2 \qquad \psi_{2\overline{z}} = -p\psi_1 \tag{16}$$

where ψ_1 and ψ_2 are complex valued functions, but $p(z,\overline{z})$ is real valued. This system leads to the surface given by

$$\begin{aligned} X^1 + iX^2 &= 2i\int_{z_0}^{z}(\overline{\psi_1}^2 \ dz' - \overline{\psi_2}^2 \ d\overline{z}') \\ X^1 - iX^2 &= 2i\int_{z_0}^{z}(\psi_2^2 \ dz' - \psi_1^2 \ d\overline{z}') \\ X^3 &= -2\int_{z_0}^{z}(\psi_2\overline{\psi_1} \ dz' + \psi_1\overline{\psi_2} \ d\overline{z}'), \end{aligned} \tag{17}$$

with mean curvature $H = \frac{p(z,\overline{z})}{|\psi_1|^2+|\psi_2|^2}$ [6]. So, if H is constant, then $p(z,\overline{z}) = H(|\psi_1|^2 + |\psi_2|^2)$. Thus Eq. 16 becomes

$$\psi_{1z} = H(|\psi_1|^2 + |\psi_2|^2)\psi_2$$
$$\psi_{2\overline{z}} = -H(|\psi_1|^2 + |\psi_2|^2)\psi_1.$$

Splitting z into real and imaginary components, $z = t+ix$, gives generalized momentum, Hamiltonian and Poisson bracket:

$$\begin{aligned} P &= \psi_{1x}\overline{\psi_2} - \overline{\psi_1}\psi_{2x} \ dx \\ H &= i(\psi_{1x}\overline{\psi_2} + \overline{\psi_1}\psi_{2x}) + \frac{1}{2}H(|\psi_1|^2 + |\psi_2|^2)^2 \\ \{F_1, F_2\} &= (\frac{\partial F_1}{\partial \psi_1}\frac{\partial F_2}{\partial \overline{\psi_2}} - \frac{\partial F_1}{\partial \psi_2}\frac{\partial F_2}{\partial \overline{\psi_1}}) - (\frac{\partial F_2}{\partial \psi_1}\frac{\partial F_1}{\partial \overline{\psi_2}} - \frac{\partial F_2}{\partial \psi_2}\frac{\partial F_1}{\partial \overline{\psi_1}}). \end{aligned} \tag{18}$$

This Possion bracket gives rise to the sympletic form

$$\xi = d\psi_1 \wedge d\overline{\psi_2} + d\overline{\psi_1} \wedge d\psi_2.$$

This system is Hamiltonian because it can be represented in the form $\psi_{1t} = \{\psi_1, H\}$, $\psi_{2t} = \{\psi_2, H\}$[6]. This is shown in the following computations:

$$\{\psi_1, H\} = \frac{\partial H}{\partial \overline{\psi_2}} \ \text{by Eq. 18}.$$

$$= i\psi_{1x} + H(|\psi_1|^2 + |\psi_2|^2)\psi_2 = \psi_{1t}.$$
$$\{\psi_2, H\} = \frac{\partial H}{\partial \overline{\psi_1}} \text{ by Eq. 18.}$$
$$= i\psi_{2x} + H(|\psi_1|^2 + |\psi_2|^2)\psi_2 = \psi_{2t}.$$

What symmetries are this Hamiltonian structure preserved under? Consider the case when p is a function of only time $p = p(t)$. Then, the system has solution

$$\psi_1 = r(t)\exp(i\lambda x) \quad \psi_2 = s(t)\exp(i\lambda x),$$

where $\lambda \in \mathbb{R} - \{0\}, r(t) = p_1 + ip2$ & $s(t) = q_1 + iq_2$ where $p1, p_2, q_1, q_2$ are real-valued functions. Then

$$H_0 = \frac{H}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)^2 - \lambda(p_1 q_1 + p_2 q_2) \text{ and}$$
$$\{F_1, F_2\} = (\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial q_1} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial q_2}) - (\frac{\partial F_2}{\partial p_1}\frac{\partial F_1}{\partial q_1} - \frac{\partial F_2}{\partial p_2}\frac{\partial F_1}{\partial q_2}).$$

Then, the Hamiltonian and Poisson structure are preserved by an S^1 action

$$\begin{bmatrix} p_1 \to p_1\cos\phi - p_2\sin\phi \\ p_2 \to p_1\sin\phi + p_2\cos\phi \end{bmatrix} \quad \begin{bmatrix} q_1 \to q_1\cos\phi - q_2\sin\phi \\ q_2 \to q_1\sin\phi + q_2\cos\phi \end{bmatrix}.$$

As

$$H_0 = \tfrac{H}{2}\left((p_1\cos\phi - p_2\sin\phi)^2 + (p_1\sin\phi + p_2\cos\phi)^2 + (q_1\cos\phi - q_2\sin\phi)^2 + (q_1\sin\phi + q_2\cos\phi)^2\right)^2$$
$$-\lambda\left((p_1\cos\phi - p_2\sin\phi)(q_1\cos\phi - q_2\sin\phi) + (p_1\sin\phi + p_2\cos\phi)(q_1\sin\phi + q_2\cos\phi)\right)$$
$$= \frac{H}{2}\left((\cos^2\phi + \sin^2\phi)(p_1^2 + p_2^2 + q_1^2 + q_2^2)\right)^2 - \lambda\left(p_1 q_1(\cos^2\phi + \sin^2\phi) + p_2 q_2(\cos^2\phi + \sin^2\phi)\right)$$
$$= \frac{H}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2)^2 - \lambda(p_1 q_1 + p_2 q_2).$$

A similar computation can be used to show that the Possion bracket is also conserved under the S^1 action. It can be verified that this system is also invariant under the transformation

$$\begin{bmatrix} X^1 \to X^1\cos\tau - X^2\sin\tau \\ X^2 \to X^1\sin\tau + X^2\cos\tau \\ X^3 \to X^3 + 4M\tau \end{bmatrix},$$

where τ is the helicoidial transformation discussed above and M is the induced metric on the space [6].

What does this example show?

In the example, the constant mean curvature surface was parametrized in Eq. 17, which looks very similar to the Weierstrass-Enneper representation given in Eq. 13. Yet, these parameterizations are path integrals, so for either to be well defined, their exterior derivative must be zero. This means for Eq. 17,

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}(\overline{\psi_1}^2) &= -\frac{\partial}{\partial z}(\overline{\psi_2}^2) \\ \frac{\partial}{\partial \bar{z}}(\psi_2^2) &= -\frac{\partial}{\partial z}(\psi_1^2) \\ \frac{\partial}{\partial \bar{z}}(\psi_2\overline{\psi_1}) &= \frac{\partial}{\partial z}(\psi_1\overline{\psi_2}). \end{aligned} \tag{19}$$

This gives a set of conserved quantities, which is the foundation of a Hamiltonian system. So, any surface given by Eq. 17, also satisfies Eq. 19 and is therefore a Hamiltonian system.

9 Conclusion

Connections between different areas of Mathematics are not presented or celebrated as much as they could be. The Weistrass-Enneper representations are one example of both how beautiful and powerful the connections between different areas in Mathematics can be, weaving together Differential Geometry, Complex Analysis and Hamiltonian Systems. All minimal surfaces can be given by the representation in Eq. 13, while all holomorphic functions can then generate a minimal surface by the same parametrization. This can be extended to the representation of Eq. 17 for a general surface, which will therefore have Hamiltonian structure satisfying Eq. 19. The duality also extends in the other direction; for a Hamiltonian system satisfying Eq. 19 will give yield to a surface via Eq. 17. Classifying when these surfaces have constant mean curvature remains an area open for exploration in Mathematics, making the Wiestrass-Enneper representations promising for future results, despite having already provided beautiful and powerful connections in Mathematics.

(Picture courtesy of PIXAR)

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