
Introduction to Bifurcations and The Hopf Bifurcation Theorem for Planar Systems

Roberto Muñoz-Alicea

Department of Mathematics
Colorado State University

munoz@math.colostate.edu

Report submitted to Prof. I. Oprea for Math 640, Fall 2011

Abstract. We introduce the concept of bifurcations in differential dynamical systems, with particular emphasis on Hopf bifurcations. Several versions of the Hopf bifurcation theorem exist. We use here a Hopf bifurcation theorem for planar systems by Gert van der Heijden, from the Department of Civil, Environmental, and Geomatic Engineering at the University College London. We provide definitions, the theorem, and a few examples, including Selkov's model.

Keywords: bifurcation, Hopf, Selkov

1 Introduction

Many, if not most differential equations depend of parameters. Depending on the values of these parameters, the qualitative behavior of a system's solutions can be quite different. Roughly speaking, we say that a differential equation system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mu)$$

has a *bifurcation* at the value $\mu = \mu_c$, if there is a change in trajectory structure as the parameter μ crosses the value μ_c . That is, there is a change in the number and/or stability of equilibria of the system at the bifurcation value (cf. [1], p. 173).

2 Basic Examples

Example 2.1 Consider the equation

$$\dot{x} = \mu - x^2.$$

If $\mu > 0$, there are two equilibria: $x^* = \pm\sqrt{\mu}$. The derivative of the right hand side is $Df(x, \mu) = -2x$. Evaluating at the fixed points we obtain the following: $Df(\sqrt{\mu}, \mu) = -2\sqrt{\mu} < 0$, which implies that the equilibrium $x^* = \sqrt{\mu}$ is stable; and $Df(-\sqrt{\mu}, \mu) = 2\sqrt{\mu} > 0$, which means that the equilibrium $x^* = -\sqrt{\mu}$ is unstable. If $\mu < 0$, there are no equilibria. When $\mu = 0$, the system has only one equilibrium point, $x^* = 0$. In this case, the equilibrium point is nonhyperbolic, since $Df(0, 0) = 0$, and we cannot use linearization to analyze its stability. A phase portrait, however, can help us in this case.

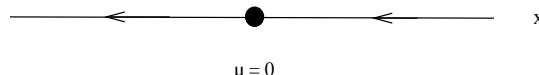


Figure 1: Phase portrait for Example 2.1

We conclude that the equilibrium point $x^* = 0$ is an unstable saddle node. This system has a *saddle-node bifurcation* at $\mu = 0$ (see Figure 2; cf. [2], p. 326).

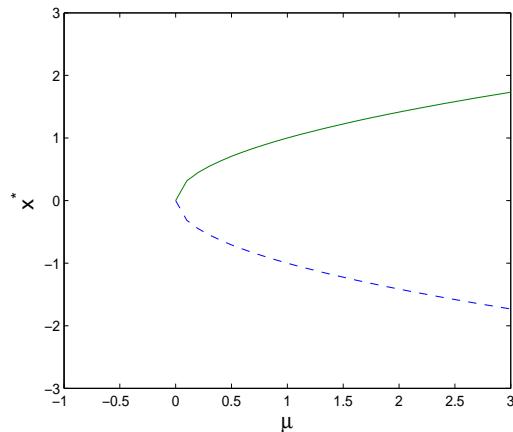


Figure 2: Bifurcation diagram for Example 2.1

Example 2.2 Consider the equation

$$\dot{x} = \mu x - x^2.$$

This equation has two equilibrium points: $x^* = 0$ and $x^* = \mu$. The derivative of the vector field is $Df(x, \mu) = \mu - 2x$. Evaluating at $x^* = 0$ we get $Df(0, \mu) = \mu$, which means that the equilibrium is stable if $\mu < 0$ and unstable if $\mu > 0$. Also, $Df(\mu, \mu) = -\mu$, which implies that $x^* = \mu$ is stable if $\mu > 0$ and unstable if $\mu < 0$. Now, if $\mu = 0$, the system has one equilibrium point, $x^* = 0$. Since $Df(0, 0) = 0$, the equilibrium $x^* = 0$ is nonhyperbolic. As before, we use a phase portrait for stability analysis. It turns out that the phase portrait for this case is the same as the one depicted in Figure 1. Thus, the equilibrium $x^* = 0$ is a saddle, hence unstable, when $\mu = 0$. In this case, we say that the system undergoes a *transcritical bifurcation* at the parameter value $\mu = 0$. The equilibria of the system exchange stability as the parameter μ crosses the value $\mu = 0$ (see Figure 3; cf. [1], pp. 173-174; [2], p.327).

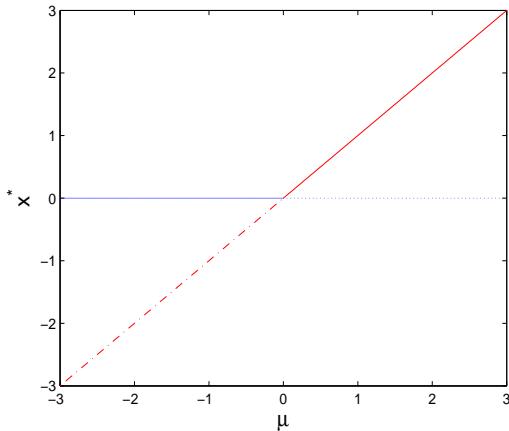


Figure 3: Bifurcation diagram for Example 2.2

Example 2.3 Consider the equation

$$\dot{x} = \mu x - x^3.$$

If $\mu > 0$, this equation has three equilibrium points: $x^* = 0$, $x^* = \pm\sqrt{\mu}$. $Df(x, \mu) = \mu - 3x^2$, so $Df(0, \mu) = \mu > 0$, which implies that the fixed point $x^* = 0$ is unstable. $Df(\pm\sqrt{\mu}, \mu) = -2\mu < 0$, so the fixed points $x^* = \pm\sqrt{\mu}$ are both stable. If $\mu < 0$, the only equilibrium point is $x^* = 0$ and $Df(0, \mu) = \mu < 0$, so the equilibrium is stable. If $\mu = 0$, again, the only equilibrium is $x^* = 0$. Since, $Df(0, 0) = 0$, the equilibrium is nonhyperbolic. As done previously, we look at the phase portrait on Figure 4:

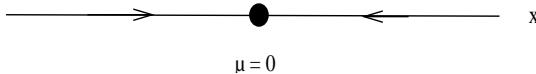


Figure 4: Phase Portrait for Example 2.3

We can see that the equilibrium in this case is stable. The system undergoes what is called a *pitchfork bifurcation* at the parameter value $\mu = 0$ (see Figure 5; cf. [2], pp. 328-329).

3 Hopf Bifurcations

A *Hopf Bifurcation* occurs when a periodic solution or limit cycle, surrounding an equilibrium point, arises or goes away as a parameter μ varies. When a stable limit cycle surrounds an unstable equilibrium point, the bifurcation is called a *supercritical Hopf bifurcation*. If the limit cycle is unstable and surrounds a stable equilibrium point, then the bifurcation is called a *subcritical Hopf bifurcation* (cf. [3], p. 264).

Before stating the theorem, we look at an example of a Hopf bifurcation on a two-dimensional system in polar coordinates (cf. [3], pp. 262-264).

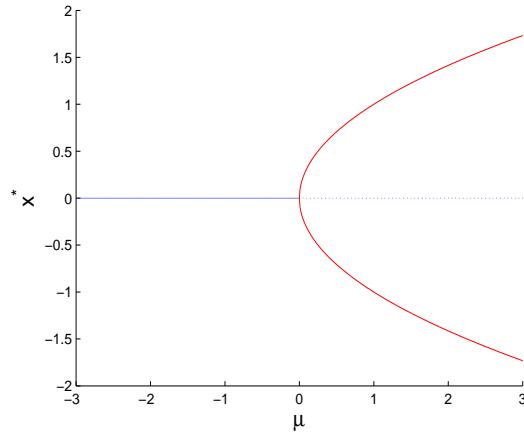
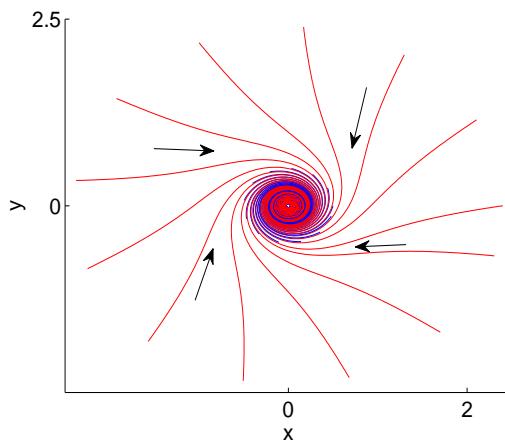


Figure 5: Bifurcation diagram for Example 2.3

Example 3.1 Consider the system

$$\begin{aligned}\dot{r} &= r(\mu - r^2), \\ \dot{\theta} &= -1, \\ r &\geq 0.\end{aligned}$$

The only critical point of this system is $r^* = 0$, that is, the origin. Since $\dot{\theta} < 0$, the trajectories move clockwise about the origin. If $\mu = 0$, then $\dot{r} = -r^3$. For nonzero r , we have $\dot{r} < 0$. Hence, there are no closed orbits and all trajectories approach the origin as $t \rightarrow \infty$. The origin is a stable focus (see Figure 6). If $\mu < 0$, then $\mu - r^2 < 0$ for all r . As in the previous case, $\dot{r} < 0$ for nonzero r values. Once more, there are no closed orbits and the origin is a stable focus (see Figure 7). Now, if $\mu > 0$, then $\dot{r} < 0$ for $r \in (\sqrt{\mu}, \infty)$ and $\dot{r} > 0$ for $r \in (0, \sqrt{\mu})$. The origin is an unstable focus and there is a stable orbit, $r = \sqrt{\mu}$ (see Figure 8). In this case, a supercritical Hopf bifurcation happens at the parameter value $\mu = 0$ (see Figure 9).

Figure 6: Phase Portrait for Example 3.1 with $\mu = 0$. The origin is a stable focus.

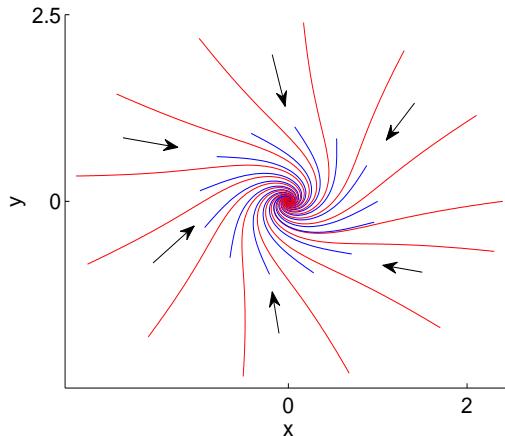


Figure 7: Phase Portrait for Example 3.1 with $\mu = -0.5$. The origin is a stable focus.

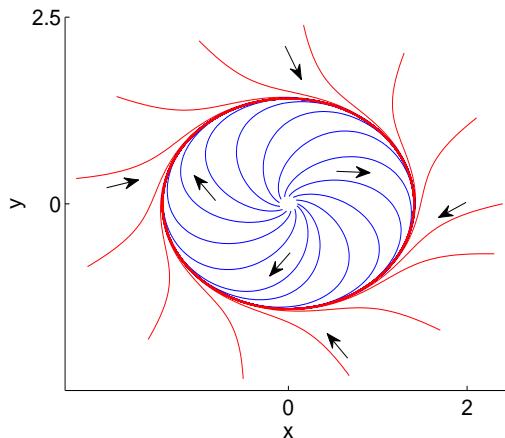


Figure 8: Phase Portrait for Example 3.1 with $\mu = 2$. The origin is an unstable focus and there is a stable orbit $r = \sqrt{\mu}$.

The following version of the Hopf Bifurcation Theorem in two dimensions, by A.A.Andronov in 1930, is stated in a paper by G.V.D. Heijden and reproduced here (cf. [4]; see also [5], pp. 299-300).

Hopf Bifurcation Theorem Consider the planar system

$$\begin{aligned}\dot{x} &= f_\mu(x, y), \\ \dot{y} &= g_\mu(x, y),\end{aligned}$$

where μ is a parameter. Suppose it has a fixed point, which without loss of generality we may assume to be located at $(x, y) = (0, 0)$. Let the eigenvalues of the linearized system about the fixed point be given by $\lambda(\mu), \bar{\lambda}(\mu) = \alpha(\mu) \pm i\beta(\mu)$. Suppose further that for a certain value of μ (which we may assume to be 0) the following conditions are satisfied:

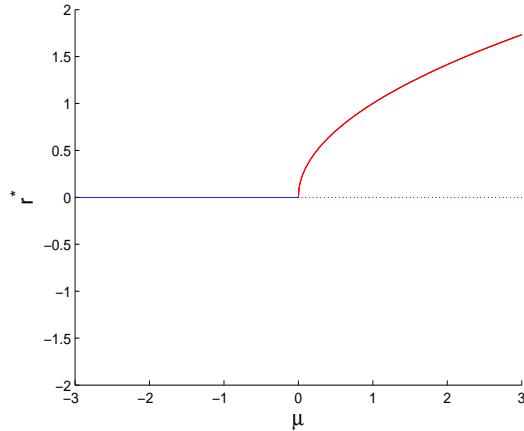


Figure 9: Bifurcation Diagram for Example 3.1: a supercritical Hopf bifurcation.

1. $\alpha(0) = 0, \beta(0) = \omega \neq 0$, where $sgn(\omega) = sgn[(\partial g_\mu / \partial x)|_{\mu=0}(0,0)]$
(non-hyperbolicity condition: conjugate pair of imaginary eigenvalues)
2. $\frac{d\alpha(\mu)}{d\mu}|_{\mu=0} = d \neq 0$
(transversality condition: the eigenvalues cross the imaginary axis with non-zero speed)
3. $a \neq 0$, where

$$a = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16\omega}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}),$$
with $f_{xy} = (\partial^2 f_\mu / \partial_x \partial_y)|_{\mu=0}(0,0)$, etc.
(genericity condition)

Then a unique curve of periodic solutions bifurcates from the origin into the region $\mu > 0$ if $ad < 0$ or $\mu < 0$ if $ad > 0$. The origin is a stable fixed point for $\mu > 0$ (resp. $\mu < 0$) and an unstable fixed point for $\mu < 0$ (resp. $\mu > 0$) if $d < 0$ (resp. $d > 0$) whilst the periodic solutions are stable (resp. unstable) if the origin is unstable (resp. stable) on the side of $\mu = 0$ where the periodic solutions exist. The amplitude of the periodic orbits grows like $\sqrt{|\mu|}$ whilst their periods tend to $2\pi/|\omega|$ as $|\mu|$ tends to zero.

Note: the condition on the sign of ω was added as a correction to the paper on 12/6/2011, after consulting with Dr. van der Heijden via electronic mail.

Example 3.2 Consider the Liénard equation

$$\ddot{x} - (\mu - x^2)\dot{x} + x = 0.$$

If we let $u = x, v = \dot{x}$, we can rewrite the equation as a two-dimensional first order system

$$\begin{aligned} \dot{u} &= v, \\ \dot{v} &= -u + (\mu - u^2)v. \end{aligned}$$

The only equilibrium point is the origin. The Jacobian matrix for the linearized system about the origin is

$$\begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}.$$

The eigenvalues of the Jacobian matrix are $\alpha(\mu) \pm \beta(\mu) = \mu/2 \pm i\sqrt{4-\mu^2}/2$. Notice that $\alpha(0) = 0$ and $\omega = \beta(0) = -1 \neq 0$. Also, $d = \frac{d\alpha(\mu)}{d\mu}|_{\mu=0} = 1/2 \neq 0$. Lastly, $a = -\frac{1}{8} \neq 0$. Hence, all the conditions of the Hopf Bifurcation Theorem are satisfied. Since, $ad = -1/16 < 0$, the origin is stable for $\mu < 0$ (see Figure 10) and unstable for $\mu > 0$, where there is a stable periodic orbit (see Figure 11). The system has a supercritical Hopf bifurcation at $\mu = 0$ (cf. [4]).

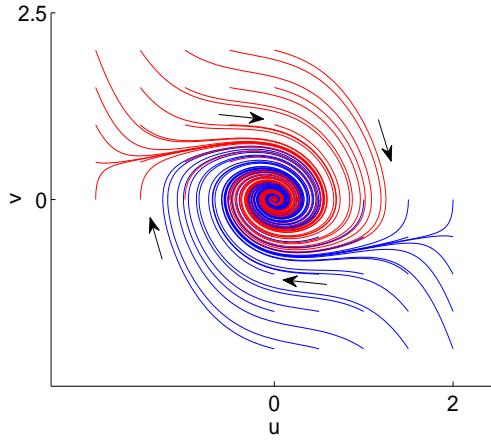


Figure 10: Phase Portrait for Example 3.2 with $\mu = -0.3$. The origin is a stable focus.

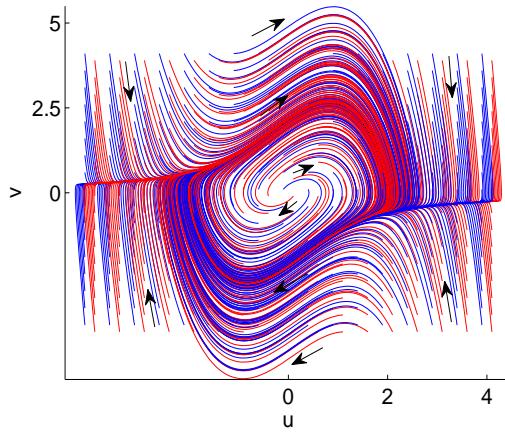


Figure 11: Phase Portrait for Example 3.2 with $\mu = 1$. The origin is an unstable focus and there is a stable periodic orbit.

Example 3.3 Consider the Sel'kov model for glycolysis, a process by which living cells break down sugar to obtain energy:

$$\begin{aligned}\dot{x} &= -x + ay + x^2y, \\ \dot{y} &= b - ay - x^2y,\end{aligned}$$

where x and y represent the concentrations of ADP and F6P, respectively, and $a, b > 0$ (cf. [6], pp. 205-209). The equilibrium for this system is

$$x^* = b, \quad y^* = \frac{b}{a+b^2}.$$

If we change coordinates by letting $\tilde{x} = x - b$ and $\tilde{y} = y - \frac{b}{a+b^2}$, then $x = \tilde{x} + b$, $y = \tilde{y} + \frac{b}{a+b^2}$, $\dot{x} = \dot{\tilde{x}}$, and $\dot{y} = \dot{\tilde{y}}$. If we then rename $\tilde{x} = x$ and $\tilde{y} = y$, the system becomes

$$\begin{aligned}\dot{x} &= -(x + b) + a(y + \frac{b}{a+b^2}) + (x + b)^2(y + \frac{b}{a+b^2}), \\ \dot{y} &= b - a(y + \frac{b}{a+b^2}) - (x + b)^2(y + \frac{b}{a+b^2}),\end{aligned}$$

where the equilibrium now is $(0, 0)$.

The Jacobian matrix for this system at the equilibrium is

$$\begin{pmatrix} -1 + \frac{2b^2}{a+b^2} & a + b^2 \\ -\frac{2b^2}{a+b^2} & -a - b^2 \end{pmatrix}$$

The eigenvalues of the Jacobian matrix are

$$\lambda_{\pm}(a, b) = \frac{a+a^2-b^2+2ab^2+b^4 \pm i\sqrt{4(a+b^2)^3-(a(1+a)+(-1+2a)b^2+b^4)^2}}{-2(a+b^2)}$$

Let

$$\alpha(a, b) = \operatorname{Re}(\lambda_{\pm}(a, b)) = \frac{a + a^2 - b^2 + 2ab^2 + b^4}{-2(a + b^2)}$$

and

$$\beta(a, b) = \operatorname{Im}(\lambda_{\pm}(a, b)) = \frac{\sqrt{4(a + b^2)^3 - (a(1 + a) + (-1 + 2a)b^2 + b^4)^2}}{-2(a + b^2)}.$$

In order for condition 1 of the Hopf Bifurcation Theorem to be satisfied, we need

$$b = b_1(a) = \sqrt{(1 - \sqrt{1 - 8a} - 2a)/2}$$

or

$$b = b_2(a) = \sqrt{(1 + \sqrt{1 - 8a} - 2a)/2}$$

The derivative of the real part of the eigenvalues with respect to the parameter b at each of these values is

$$\frac{d\alpha(a, b)}{db} \Big|_{b=b_1(a)} = \frac{\sqrt{2 - 16a}\sqrt{1 - \sqrt{1 - 8a} - 2a}}{1 - \sqrt{1 - 8a}},$$

and

$$\frac{d\alpha(a, b)}{db} \Big|_{b=b_2(a)} = \frac{\sqrt{2 - 16a}\sqrt{1 + \sqrt{1 - 8a} - 2a}}{-1 - \sqrt{1 - 8a}}.$$

The expression for the quantity on condition 3 of the Hopf Bifurcation Theorem is

$$\left(-\frac{1}{8} - \frac{(a+b)^2(\frac{b}{a+b^2} + y)^2}{2\sqrt{4(a+b^2)^3 - (a(1+a) + (-1+2a)b^2 + b^4)^2}} \right) \Big|_{b=b_1, b_2}(0, 0).$$

Let us now fix the parameter value $a = 0.1$. Then $b_1(a) = b_1(0.1) \approx 0.419992$, and $b_2(a) = b_2(0.1) \approx 0.789688$. Also, $\omega_1 = \beta(0.1, b_1(0.1)) \approx -0.525731$, $\omega_2 = \beta(0.1, b_2(0.1)) \approx -0.850651$, $d_1 = \frac{d\alpha(0.1, b)}{db} \Big|_{b=b_1(0.1)} \approx 0.679561$, and $d_2 = \frac{d\alpha(0.1, b)}{db} \Big|_{b=b_2(0.1)} \approx -0.488054$. Lastly, the values for the expression on condition 3 of the theorem are $a_1 \approx -1.223$ and $a_2 \approx -0.475021$.

Thus, for both values of the parameter b , the Hopf Bifurcation Theorem applies. In each of the two cases, $b = b_1 \approx 0.419992$ and $b = b_2 \approx 0.789688$, there is a Hopf bifurcation. We have $d_1 > 0$ and $a_1 d_1 < 0$. Hence, the origin is stable for $b < b_1$, and there is a stable limit cycle for $b > b_1$, provided b is sufficiently small. In addition, $d_2 < 0$ and $a_2 d_2 > 0$. Thus, the origin is stable in the region $b > b_2$, and there is a stable limit cycle in the region $b < b_2$. Therefore, the origin is unstable in the region $b_1 < b < b_2$, where there exists a unique and stable limit cycle. (see Figures 12, 13, and 14).

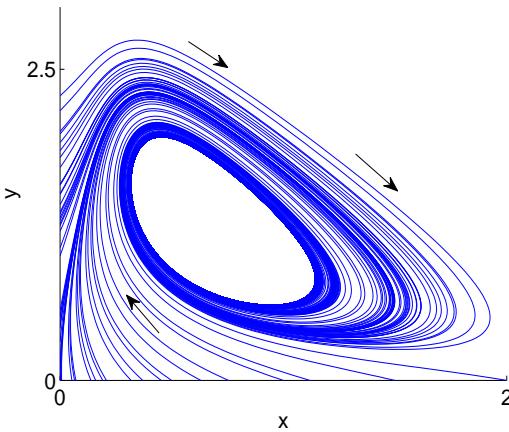


Figure 12: Phase Portrait for Example 3.3 with $b = 0.6$. The origin is an unstable focus and there is a stable periodic orbit.

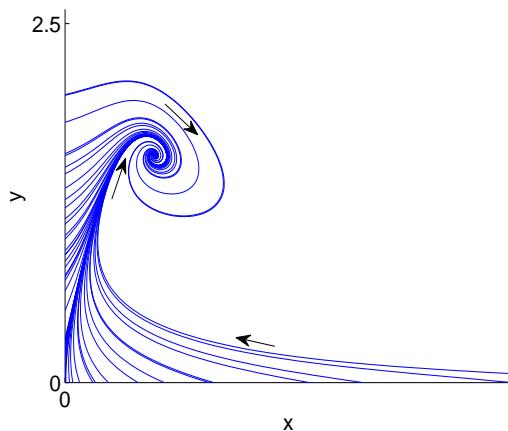


Figure 13: Phase Portrait for Example 3.3 with $b = 0.3$. The origin is a stable focus.

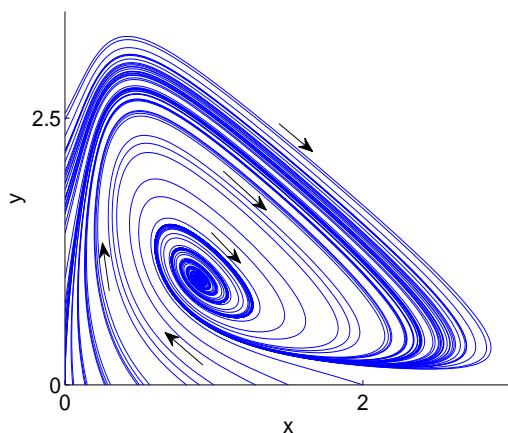


Figure 14: Phase Portrait for Example 3.3 with $b = 0.9$. The origin is a stable focus.

4 Summary

Bifurcation theory is a vast and active area of research. We have explored, through a few simple examples, some of the different types of bifurcations on one and two-dimensional ordinary differential equations systems. We have analyzed the bifurcation examples through phase portraits, bifurcation diagrams, and the Hopf Bifurcation Theorem. In order to further our understanding of bifurcations, it is recommendable to understand the normal form of systems of differential equations, stability, chaos, and bifurcation theory for systems of more than two equations, and even numerical analysis of these systems.

References

- [1] Verhulst, F., *Nonlinear Differential Equations and Dynamical Systems*, 2nd edition, Springer, Berlin (1996).
- [2] Perko,L., *Differential Equations and Dynamical Systems*, 2nd edition, Springer, New York (1996).
- [3] Lynch, S., *Dynamical Systems With Applications Using Matlab*, Birkhauser, Boston (2004).
- [4] van der Heijden, G., *Hopf Bifurcation*, <http://www.ucl.ac.uk/~ucesgvd/hopf.pdf>
- [5] Meiss, J.D., *Differential Dynamical Systems*, SIAM, Philadelphia, PA (2007).
- [6] Strogatz, S.H., *Nonlinear Dynamics and Chaos*, Perseus Book Publishing, LLC, Cambridge, MA (1994).