

When is a Curve a Line?

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Abstract. In this paper I want to extend some of the basic concepts we have studied in this class into the realm of Riemannian geometry. The point of this is to jump off from the idea of Gauss' Theorema Egregium into the modern world of Riemannian geometry and give at least one example of its power in the realm of physics. We will define a manifold, differential structures, tangent bundles, and affine connections. Unfortunately we will then only discuss the idea of geodesics and its role in the realm of physics.

Keywords: Differential structure, Tangent Bundle, Connections, Geodesics

1 Introduction

Suppose you are a 16th century explorer and you want to travel the world while making note of where you have been, what you have seen, and all the while keeping track of where it was that you saw certain things, i.e. 'Here there be monsters... but over there the islanders love to party.' Being a kind soul, you want to make sure that other explorers are not forced into the same hardships you have endured throughout your gallant forays into the unknown. So you make a chart of this region of the world and that region of the world. It would make absolutely no sense whatsoever to encode the places you have been on a sheet of parchment the same size as the swath of the globe you are trying to depict (if it is a globe we live on). Not even a factor of a hundred would really be small enough to make it manageable. Even if

you just wanted to chart one hundred square kilometers, you would need to bring it down a factor of 10^{-6} just to fit it on a one meter square bit of paper. Keep in mind that 100 [km^2] is roughly a quarter the size of Colorado.

BUT WAIT!!!

What about the intersection going from one chart to an adjacent one? How are we going to manage that? We are going to have to include a little bit of what precise mathematicians call “fluff” all around the borders of our charts. Now, to a mathematician, wherever there is “fluff” there is potential for monsters. What if we use a certain kind of “fluff” on this chart and a different type of “fluff” on the other? Will our progeny be able to navigate the world or will they be trapped in the horrendous borderlands of our imprecise “fluff”? This is of great worry to a caring and benevolent explorer like yourself. To get around this we will have to answer a few questions like what kind of space is our world, what are paths in our world, and how are we to navigate in the regions of “fluff”? ¹

The next thing we will need to consider is ‘what exactly do we mean by local coordinates?’ We want to be able to tell our future explorers what direction to look in order to see something from somewhere else or even more importantly — if I am at point a and want to get to point b which direction should I go? Which direction will I find Sirius? What is the fastest route out of Hades? This sort of information will be convenient to have encoded on our charts in some form or fashion that is easily translated into physical awareness using compasses, astronomy, prevailing winds, or, if nothing else, a six sided die. I am not entirely sure how exactly we would navigate in our world with a six sided die, but surely there is a hypothetical universe in which this could be completely sound practice.

Once we have a sense of direction we are happy, right? Wrong. We constantly need to recalibrate along our journey’s paths. That is the problem with local coordinates, they only work in a local region. Here is where we will have to deal with one of the potentia problems of the proverbial “fluff.” We want to not only know where we are in the “fluff,” but what direction is toward our next chart or charts. Surely, you must be saying to yourself, one could just continue on in a straight line into the next chart and close your eyes while amidst the “fluff.” Well that sounds like a great idea; I really like that idea. Just one problem — what is a straight line? You and I both know what we mean by a straight line, but the confusion comes in when we try to compare my straight lines to yours. Consider the example of a boat sailing on the ocean with a perfectly north heading from a point on the equator at longitude 14 degrees. Consider a fellow traveller in a similarly oriented boat sailing with the same heading yet he is located at 18 degrees longitude. They both continue in what appears to be a straight line. Now what if you are watching this from the international space station, if such a station exists? (depending on what century we are imagining at the moment) Your perspective grants you the ability to notice that they are not at all going in straight lines, they are going along an arc of a circle concentric with the earth and sharing the same radius. This is simply anecdotal; but it does give the flavor that we need to be precise by what we mean by going straight.

¹Throughout this essay I will be taking the philosophical stance that geometry is about the world around us. It is perfectly sound to study geometrical objects in and of themselves without any consideration for the universe, but that is not the approach taken here.

Here we need to have a different kind of translation from chart to chart in terms of directions as well as locations. Once we have done this, we can then talk about what it means to go “straight” and how we can check a given path for its inherent “straightness.”

In Euclidean space, going straight means just that. The problem with our example is this whole curvature of the surface of the earth. There is a similar problem with the universe as a whole. There exists curvature in the fabric of space-time. This curvature gives rise to one of the most geometric of physical theories, Einstein’s General Relativity. In Einstein’s formulation of the governing equations, there is a magical object called the “stress-energy tensor” which tells space-time how to curve. Space-time, being a jovial conversational partner, in turn tells particles how to move. This lovely colloquium of fields and motions is a very common construct in physical theories. However, in this particular setting the theory lends itself to a very geometrical expression.

2 The Space of our World

2.1 Manifolds

What is the shape of our space? What is a technical definition of our space? These are difficult questions to answer, ask any string theorist or cosmologist, but the consensus is that no matter which version of this particular type of object, our world is in fact a manifold.

Definition: We say a *manifold* M to mean a topological space which fulfills the following three conditions:

- (i) M is locally Euclidean of dimension n
- (ii) M is Hausdorff
- (iii) M is second countable

So lets dwell on these conditions for a moment just so we are all on the same page. What it means to be locally Euclidean is that every point $p \in M$ has a neighborhood that is homeomorphic to a neighborhood of \mathbb{R}^n . You will see in what follows is that this condition is made for a few reasons, two of which are that what we experience as space is very much like \mathbb{R}^3 , and since we know how to do analysis (calculus) on \mathbb{R}^n we are also being hopeful about this so that we can bring to bear the tools of analysis on the spaces we choose to study.

The condition of the space being Hausdorff means that we have enough open sets that are small enough so that we can separate any two distinct points in M into two disjoint neighborhoods. This condition can be thought of as our space being dense in points.

The last condition of second countability ensures us that we can talk about sequences

instead of nets. Intuitively there is barely even a subtle difference between the two, but mathematically the difference is significant despite the subtlety.

Assuming our universe is indeed a manifold, there is one more thing that is silly but we might as well impose regardless. We will assume that our universe is connected. (If it is not we wouldn't really be able to tell since we are part of this connected component and therefore we [physicists] would not care) Since we are making this assumption and imposing the condition that our manifolds are connected, it is easy to see that the function $\underline{\dim} : M \rightarrow \mathbb{N}$ telling us the local dimension of our manifold is a constant function. So here we are with a space that we would really like to use calculus on. What follows in the sequel is the additional structure necessary for the use of such powerful tools.

2.2 Differentiable Structures

What does it mean for a function, $f : M \rightarrow \mathbb{R}$ to be differentiable? Well just so you will have a clearer idea of what I am talking about in the following few sections let's write down a simple definition.

Definition: We say a *homeomorphism* to mean a continuous bijective function with a continuous inverse.

A homeomorphism is the isomorphism in the category of topological spaces. It is a way of saying when two topological spaces are the “same” topologically. Since topological spaces come equipped only with the structure of open sets, a homeomorphism is a less stringent condition than a diffeomorphism. Any manifold comes equipped with homeomorphisms to and from local neighborhoods and \mathbb{R}^n , so it would make sense to just require the composition function $f \circ \psi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}$ to be differentiable at p when ψ is a homeomorphism from a neighborhood V of p . This is great, but does it depend on our choice of ψ ? Well here is just one way that it might. What if ψ is not differentiable or does not have a differentiable inverse, how can the composition then be differentiable? So if we are going to rely on these maps at all, it might be a good idea to put a few more restrictions on them.

One in particular is the fact that not all homeomorphisms are also diffeomorphisms (despite there existing a diffeomorphism corresponding to any homeomorphism of a subset of $M = \{\mathbb{R}^n \mid n \leq 3\}$). That is fine though we can just restrict our homeomorphisms to diffeomorphisms and forget about the pathological homeomorphisms. This is a bit stronger than we really need but it will be ok for now while we build up to an appropriate definition. But what about the choice of neighborhood. Here we have our first encounter with the “fluff” monsters. What if we choose a different neighborhood, say U , and a different diffeomorphism, say φ . Well if we do stuff like that we need for the function $\psi \circ \varphi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to be such that $\psi \circ \varphi^{-1} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and this is guaranteed up to i when the coordinate maps are at least in $C^i(\mathbb{R}^n, \mathbb{R}^n)$. So we are really getting more than we need from requiring the maps to be diffeomorphisms. When we say that our coordinate maps are diffeomorphisms, this implies $i \geq 1$. But we want our functions to be $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ just so we do not need to worry about

how many derivatives we take. So we do not really need the diffeomorphic condition on our functions. What we do need is simply $\psi \circ \varphi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to be such that $\psi \circ \varphi^{-1} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$

Lastly we want to make sure that we have all the charts there could be. This is built into the idea of a maximal atlas. An atlas is just what you want it to be, a collection of charts. A maximal atlas is stricter condition that guarantees that no matter who you are and what chart you bring me, if your chart agrees with all of my charts, I already have it in my atlas.

Definition: We say a *smooth n-manifold* to mean that M is an n -dimensional manifold with a set of injective mappings $\alpha : M \supset U_\alpha \rightarrow \mathbb{R}^n$ With U_α is open in M and the following three conditions hold.

- (i) $\bigcup_\alpha U_\alpha = M$
- (ii) For every V_ψ, U_φ such that $V_\psi \cap U_\varphi \neq \emptyset$ the corresponding transition function $\psi \circ \varphi^{-1} : \mathbb{R}^n \supset \varphi(U) \rightarrow \psi(V) \subset \mathbb{R}^n$ is an element of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and is a diffeomorphism
- (iii) The collection of injective mappings is called an atlas and is maximal with respect to (i) and (ii)

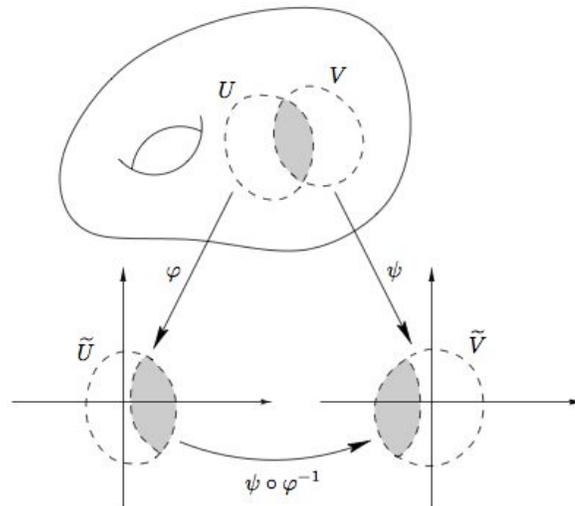


Figure 1: Transition Function {image from Lee’s smooth manifolds text}

We would like to address the idea of *maximal* atlases. So let an atlas, \mathcal{A} , be called maximal when given any chart ξ that is compatible with all charts contained in \mathcal{A} , ξ is already contained in \mathcal{A} . Some authors call this object a complete atlas. What it means is exactly what we want. Whenever someone comes up with a chart of our space and they say it is compatible with our charts, we say “no thank you we already have that one right here.” We will from now on refer to a smooth manifold as a pair (M, \mathcal{A}) where M is a manifold and \mathcal{A}

is a maximal atlas. This also allows us to give a definition of diffeomorphism with respect to the atlas.

Definition: We say $f : M \rightarrow N$ is a diffeomorphism of smooth manifolds (M, \mathcal{A}) and (N, \mathcal{B}) to mean that $\psi \in \mathcal{B}$ if and only if $\psi \circ f \in \mathcal{A}$.

We now can expect a function f on our manifold (M, \mathcal{A}) to be a differentiable function if its composition with any coordinate chart's inverse is a differentiable function, i.e. f is differentiable at p if $(f \circ \psi^{-1})|_p \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Throughout the remainder of our time together I will be implying "smooth" whenever I speak of a manifold, unless specifically stating otherwise.

Another way of thinking about these smooth structures is in terms of coordinate neighborhoods. Since we are equipping our manifold with maps to \mathbb{R}^n we will have local coordinates imposed on our manifold from the inverse maps. The smooth structure guarantees that our coordinate neighborhoods will be compatible on the intersection of two different neighborhoods.

WARNING : There is nothing forbidding two diffeomorphic smooth manifolds from having incompatible smooth structures. What this means is that there is not always one unique maximal atlas that gives rise to the "same" manifold.

2.3 Curves

We conclude this section with a brief definition of what we mean by a curve on a manifold. The definition is rather straightforward; the motivation, not so obvious. We are interested in curves, at least in the terms of our previous analogy, because they are the paths of "particles," or a ship exploring the planet, or you as you explore the universe. The paths are parametrized by time, our curves will be parametrized by t . We will use curves in the next section to get at tangent spaces, but our main concern for curves will be concerns of efficiency with respect to length of our journey or minimization of the navigational corrections necessary (I like going straight towards my goal despite your impressions of that fact as ascertained through this paper).

Definition: We say a curve to mean a function $\gamma : I \rightarrow (M, \mathcal{A})$ defined on an interval I such that $\psi \circ \gamma : I \rightarrow \mathbb{R}^n$ is a path in the normal sense in \mathbb{R}^n .

We do not, at this time, put any other restrictions on what we call curves.

3 Tangent Space

3.1 Bundles

Before we define what it is we mean by a tangent space, which I am assuming you have a relatively clear idea of already, we will discuss something a bit more general and then see how the tangent space is a specific example of a fiber of a bundle. There are many aspects of bundles that lead to different intuitive insights into their structure. As an object they consist of simply two smooth manifolds and a projection down to the base manifold, typically the manifold one is studying. There are many different types of bundles one can put on a given manifold. The prevalent bundles are all easily thought of as vector bundles, so we will give the formal definition below.

Definition: We say a *vector bundle* to mean a pair of manifolds E and M together with a surjective map $\pi : E \rightarrow M$ satisfying the following three conditions.

- (i) Each fiber $E_p \triangleq \pi^{-1}(p)$, where $\pi^{-1}(p)$ is the pre-image of p under π , has the structure of a vector space.
- (ii) For every $p \in M$, there exists a neighborhood U of p and a diffeomorphism $\varphi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ called the local trivialization of E , such that the following diagram commutes (π_1 is the projection onto the first factor of the product space).

$$\begin{array}{ccc}
 \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{R}^n \\
 \pi \downarrow & & \downarrow \pi_1 \\
 U & = & U
 \end{array}$$

- (iii) The restriction of φ to any fiber $\varphi : E_p \rightarrow \{p\} \times \mathbb{R}^n$ is a isomorphism of vector spaces.

The formal definition of a vector bundle lets us rephrase what we think of as the tangent bundle into this language. The tangent bundle is simply the space we get by “glueing” the tangent space of each point of our manifold together. The proper term for this glueing is that the tangent bundle TM is the *disjoint union* of tangent spaces T_pM for all $p \in M$. The TM as a set is simply the product $M \times \mathbb{R}^n$. We can now see that to put a vector field on our manifold we can think of the vector field X as a function $X : M \rightarrow TM$ that takes a point on the manifold and assigns a corresponding vector. It will be a bit of work, but not too much, to extend this idea to a smooth(differentiable) vector field using the definition we wrote down earlier for a smooth function on M .

The first criterion of the definition says that the pre-image of p under π has the “structure of a vector space.” The remaining two only say that the “vector space” associated with the fiber of a point must be isomorphic as a vector space to \mathbb{R}^n . This is one of the places where math diverges from the normal; in some cases we do care about the difference between two isomorphic spaces. Two bundles that are isomorphic, but we imagine as separate entities, are the tangent and cotangent bundles (in most literature it is denoted T^*M or at a point it will be T_p^*M). I will not be going into what the cotangent bundle is explicitly, but at least one way to think of it is as the dual space to the tangent bundle, TM .

WARNING: aside on topology

When we think of the tangent bundle in this manner, as a product set, we can say that locally TM has the product topology as a result of condition (ii) of our definition. Just a caveat though, the tangent bundle of a manifold does not at all have to have the same topology as the product topology on $M \times \mathbb{R}^n$. But there are some interesting spaces that have what are called trivial bundles, e.g. $M = \{S^n \mid n \text{ is odd}\}$.

3.2 The Tangent Space, T_pM

The point of putting a vector field on a manifold will become more clear once we have an idea of what a connection is. But for the time being we will focus on defining what the tangent space at point will be. But first, what do we want it to be? If the gods are generous, the tangent space of a manifold at a point will be the best linear approximation of $M|_p$.

We also want it to line up with our definitions from class which were twofold. The one I will be using is the definition that relies on curves. To construct a tangent vector at a point p , I will use some curve $\gamma(t)$ and $\gamma(0) = p$. The first thing we would want to do is just take a derivative of γ at 0.

This works, sort of. The only problem is that this result is not necessarily in our manifold. So we have a function from the interval to our manifold whose derivative is not an element of either. Where does that derivative live? We will basically build a space out of all of these “derivatives.” Where will this space be located? First let’s reconsider this curve and compose it with a local chart ζ , i.e. $\zeta \circ \gamma : \mathbb{R} \supset I \rightarrow V \subset \mathbb{R}^n$. The derivative of this function will be a vector in \mathbb{R}^n and that is exactly what we want to use in building the tangent space for our mind’s eye. One canonical way of doing this is with the unit speed coordinate lines in \mathbb{R}^n and mapping this line onto a point $p \in M$. Taking derivatives, each coordinate line gives a unit “tangent vector” in a linearly independent direction, the span of which will give you a copy of \mathbb{R}^n as desired and it is located, at least figuratively, with its origin at $p \in M$.

This is fine for how we think of tangent vectors. An equivalent definition of them is the actual differential operators that we evaluate at a point to get the vectors we just discussed. This will come back later, but for now let’s just hang on to the more concrete geometric objects we are used to.

Let's go towards an analogy that may help with one way of thinking about the tangent space. Since what we care about is the magnitude and direction ultimately (since that is what a vector is according to a freshman physics major), we want to define a tangent space at a point to be something that no matter what coordinate neighborhood we choose, we will agree on the magnitude and direction. To do this, I invite you to consider a pair of ghost penguins skating around near the north pole. (why they are ghosts will become clear momentarily) Each of them will skate through the north pole along different paths simultaneously. We also have a pair of astronauts in their own spaceships located at least a great distance from each other. They each have their own interpretation of where the penguins are and what direction and magnitude their velocities are. But at the point of the north pole, when the two penguins pass through one another, the astronauts will agree on one thing, whether or not the penguins had the same velocities at that moment.

This velocity, or equivalence class of velocities is a tangent vector. We will take as our T_pM all possible equivalence classes of velocities to curves passing through p .² The only problem with this idea of the tangent space is that it seems like it may depend on the ambient space that our manifold lives in. You are right to be concerned, but the tangent space just constructed really does not depend on that ambient space at all. This is apparent from our earlier construction of T_pM . To convince yourself of this, simply remember that we can use our coordinate neighborhoods and the fact that they are compatible to translate what one astronaut observes into what the other records.

4 Affine Connections

4.1 Motivation

We want to begin this section with a question, “what does it mean to differentiate a function on a manifold?” In regular old calculus, if we have a function that depends on multiple parameters (coordinates) it changes at different rates depending on what direction one varies the point of interest. We saw this with the gradient and directional derivatives in multivariable calc. We have already indirectly assumed that we can take derivatives, that is how we built tangent vectors. But another thing we learned in calculus is how to take derivatives of vector valued functions. So what now if I want to differentiate such a thing on my manifold in any given direction I choose from the tangent space? We ultimately are going to want to differentiate the vector valued function that at each point along a curve gives us the curve's tangent vector.

A problem shows up when we try to write a derivative in a difference quotient with tangent vectors as the terms and then take a limit. The tangent vectors in the difference quotient live in different tangent spaces. How do we subtract two objects from different domains in a natural way? Some how we want to “connect” up the tangent spaces. To do this we need some sort of rule that allows us to take directional derivatives of different vector

²The ghost penguin analogy is from Mike Alder's notes for his Differential Geometry course at UWA.

fields.

In order to have an extrinsic idea of what this rule allows us to do, we consider our manifold embedded in some ambient space. We take a directional derivative of some function, vector or scalar valued, and we get a vector or scalar in return. We are more interested in the vector valued function so let's consider that case. The result is some new vector field defined on our curve. This vector field does not have to lie in our tangent space, consider an arbitrary unit speed curve on $S^2 \subset \mathbb{R}^3$ and the vector field defined by the tangent of the curve at every point. There is no a priori reason that the change in the direction of the tangent vector will be in the tangent plane alone. The case of the equator as the curves path is a perfectly valid example of a curve whose tangent field has no change in any tangent direction. So if we are to come up with some new rule to differentiate vector fields we would want to stay within the vector spaces we have built into our manifolds already; we do not want to rely on some embedding. So we can take an extra step in our extrinsic example and simply project whatever our result is down (orthogonally) into the tangent space at the point of consideration.

4.2 Definitions

There are more general definitions of a “connection”, but what we care about are vector fields on our manifold made by assigning a tangent vector to each point. We do this because we do not have to worry about any embedded space, this desire to do things intrinsically is motivated by the cruel fact of life (so far) that we cannot get outside of the universe and therefore must study its geometry in a fashion similar to ants on the surface of an apple. So we will only need the particular case of affine (sometimes called linear) connections. It is possible define a connection with respect to any bundle on M , but we will focus here on the shortest path to what we want, “straight” lines.

Definition: Let $\mathcal{T}(M)$ be the space of all vector fields on M . We say an *affine connection* to mean a map $\nabla : \mathcal{T}(M) \times \mathcal{T}(M) \rightarrow \mathcal{T}(M)$ that satisfies the following conditions. $\{\nabla(X, Y) \mapsto \nabla_X Y\}$

- (i) $\nabla_X Y$ is linear over $C^\infty(M)$ in X . This means that $\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$ for all $g, f \in C^\infty(M)$
- (ii) $\nabla_X Y$ is linear over \mathbb{R} in Y . This means that $\nabla_X (aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$ for all $a, b \in \mathbb{R}$
- (iii) ∇ has its own version of the Leibniz rule. (product rule) $\nabla_X (fY) = f\nabla_X Y + (Xf)Y$ for all $f \in C^\infty(M)$

Since what we care about is that this idea of a connection is a local operator we should check that this is satisfied, i.e. that we only care about X and Y in a local neighborhood to define $\nabla_X Y$.

Lemma: Let ∇ be a connection on M , $X, Y \in \mathcal{T}(M)$ and $p \in M$, then $\nabla_X Y|_p$ only depends on the values of X and Y in an arbitrarily small neighborhood of p .

Proof: First it will be beneficial to state the lemma in a different way.

RESTATEMENT: If in an arbitrarily small neighborhood of p , $\tilde{X} = X$ and $\tilde{Y} = Y$ then $\nabla_{\tilde{X}} \tilde{Y}|_p = \nabla_X Y|_p$.

If this restatement of the lemma is true, so is the lemma.

So let's replace the field Y with $Z = Y - \tilde{Y}$. Now if we can show that $\nabla_X Z|_p = 0$ on a neighborhood $U \ni p$ we will have shown that $\nabla_X Z|_p$ only depends on the values of Z in the neighborhood and Z can be whatever it wants to be outside that neighborhood.

Consider a partition of unity on M and choose φ_U , the function on U such that $\varphi(p) = 1$ and $\varphi(U^c) = 0$. This will give us $\varphi Z \equiv 0$ for all of M because $Z \equiv 0$ on the support of φ and $\varphi \equiv 0$ on the possible support of Z .

So we have φZ a constant function, therefore $\nabla_X(\varphi Z) = \nabla_X Z + (X\varphi)Z = 0$. The second term is identically zero on all of M because of what we stated above. Therefore $\nabla_X Z = 0$ for all of M , but especially for $\nabla_X Z|_p$. So as far as Z is concerned we only care about it locally.

The argument for only local dependence on X follows a similar line of reasoning and ends with $\nabla_{\varphi Z} Y = \nabla_{0 \cdot \varphi Z} Y = 0 \cdot \varphi \nabla_Z Y = 0$ where in this case $Z = X - \tilde{X}$. This is analogous to taking the directional derivative of any vector valued function in the $\vec{0}$ direction.

□

This map on a manifold, when it exists (which is always), depends only on the local values of our two vector fields. We can restrict this further if we really wanted to and show that it only depends on Y locally and X at the point p . But I will leave that to you.

4.3 Restriction to Curves

We are almost there. Now that we have a connection, whatever that is, we can start thinking of ways to restrict it to a vector field defined on a curve. I say “whatever that is” because it is not at all clear from the definition exactly what this monster is going to do for us. Unfortunately, such is the case with much of modern mathematics, the definitions are not a priori accessible; we are forced into one of two methods, possibly others exist that I am unaware of. Either we simply begin to use the object on examples and attempt to understand what it is via the processes it is involved in, or we examine a different type of process, the historical one, in which the modern definition was originally cast and remolded many times until it took its modern form (which may or may not be its final form). The idea of a vector field restricted to a curve should be pretty straight forward, if what we claimed above about

the connection only relying on X at a point is true (it is). We only need to know the vector $v \in X$ at a point. So if we have some vector field X we should be able to restrict to some curve and use it to differentiate another field V along our curve.

We need at least one more idea and one more definition before we can state what we mean by this way of restricting our connection to a curve. Since ultimately what we want are geodesics, we will want to be able to take the vector field along our curve to be the tangent vectors of our curve at any given $t \in I$, and we will want to see how they change in that direction. The basic idea is that if tangent vectors do not change in the tangent direction, we are going “straight”. Another way of saying this is one of the ways we described it in class. We want a curve that has “no perceived acceleration.” If we imagine ourselves being on/in the manifold and moving along our curve, we will have some perception of whether or not we are going straight. I will come back to this first person perspective when we discuss reference frames and general relativity.

Now for another definition. Given a curve, we know how to compute its tangent vectors for all $t \in I$. Since this is the vector field we want to deal with, I will not discuss other vector fields that may be defined on our curve of interest.

Definitions: (a) We say a vector field $V(t)$ on a curve is *extensible* to mean that there exists a neighborhood U of the image of the curve $\gamma \in M$ and a vector field \tilde{V} , a smooth vector field on U , such that $V(t) = \tilde{V}\{\gamma(t)\}$. In particular we want to put the restriction on our curves and call them *extensible curves*.

(b) We say an *extensible curve* to mean that if there exists a point $p \in M$ such that for distinct t_0 and t_1 $\gamma(t_0) = \gamma(t_1)$ then for any $n \in \mathbb{N}$ we have an equality of the n^{th} derivatives $\gamma^{(n)}(t_0) = \gamma^{(n)}(t_1)$.

We add on the last restriction so that the vector field we are interested, $\dot{\gamma}(p \in M)$, in is well defined and extensible to a local neighborhood.

And now the moment we have all been waiting for, the tool that tells us when we are going straight.

Definition: Let ∇ be a linear connection on M . For each extensible curve $\gamma : I \rightarrow M$, ∇ determines a unique operator $D_t : \mathcal{T}(\gamma) \rightarrow \mathcal{T}(\gamma)$ which satisfies the following properties:

- (i) Linearity over \mathbb{R} , i.e. $D_t(aV + bW) = aD_tV + bD_tW$ for all $a, b \in \mathbb{R}$.
- (ii) Product rule, i.e. $D_t(fV) = \dot{f}V + fD_tV$ for all $f \in C^\infty(I)$.
- (iii) For any extension \tilde{V} of $V(t)$, a vector field on a curve γ , $D_tV(t) = \nabla_{\dot{\gamma}(t)}\tilde{V}$.

For any $V \in \mathcal{T}(\gamma)$, D_tV is called the *covariant derivative* of V along γ .

Now we have all the machinery and structure to finally say what the objects we are interested in are in the language of Riemannian Geometry. It should be noted that this has

been a massively tedious and difficult undertaking just to construct these precise definitions so that we can say when a curve is a line. These few pages really do not do justice to the gargantuan amount of intellectual effort and creativity that was necessary for the modern form of these objects. I just wanted to take a moment and point out that if Newton stood on the shoulders of giants, at this point we are standing on the shoulders of a stack of at least two Titans; their names are Gauss and Riemann.

4.4 FINALLY Geodesics

Without further ado, I give you the criterion for when a curve is a line:

Definition: An extensible curve γ is called a *geodesic* with respect to ∇ if its acceleration is zero, this is the case exactly when $D_t\dot{\gamma} \equiv 0$.

According to the mathematicians that I have been in contact with, this is the nicest kind of mathematics. First one does all of the hard work by understanding the definitions and notation. Then profound theorems are easily proven by unwinding the definitions and checking that the objects they concern are in correspondence with the ones we are familiar with from simple examples. So long as this correspondence holds we can use our tools to study things that are not at all familiar to us in ways that we do study familiar objects. In doing so, we still have something profound to say about such estranged objects. Such is the power and beauty of mathematics and in this particular case, geometry.

5 Physics

5.1 General Relativity

Here is where we encounter the real world which all of our hard work comes back in the hope of having tools to describe our world. Although gravity, by comparison, is the weakest of the four “fundamental” forces described by modern physicists, it is the most familiar to beings of our scale. Electromagnetism does act in ways that are ‘noticeable,’ but its nature is not as internalized in our experiences so much as the idea of rigid objects. (the fact that one ‘rigid’ object does not easily pass through another is a manifestation of the electromagnetic force) The weak and strong forces are dominant on scales barely imaginable to humans.

Our brains are arguably hard wired to understand and manipulate the effects of gravity. Humans can throw objects with a remarkably high degree of accuracy which could have given us one of many evolutionary advantages in prehistory. We are also absolutely obsessed with the idea of parabolic trajectory, e.g. (i) one of the oldest sports has many forms that fall under the category of marksmanship, (ii) the silly game “Angry Birds” is obsessive for millions upon millions of people and the whole game is about launching little birds along parabolic trajectories using a makeshift kind of slingshot.

It may seem silly to talk about parabolic paths after we have spent so much time figuring out when lines are straight. But these lines are straight in the appropriate reference frame. The brilliance of Einstein's theory is manifest in its simple formulation and conceptual structure. Einstein decided to think of gravity as a "geometrical" force. He came up with a way to imagine particles moving through space on what are straight lines determined by a curvature of space-time. Let's return to our visualization of two boats sailing "north." We have two paths that we know are not "straight" from the point of view of our astronaut. However from the point of view of the two captains, (who naïvely think that their world is flat) are going "straight," yet some "force" is acting that pulls them closer, so long as they are moving in the northern direction. If we rephrase the analogy in terms of ants crawling on an apple, we avoid the idea of stupid ship captains. Regardless there is something going on that you and I do not call a force. The "attraction" is a result of the curvature of the earth/apple. One may think that this will debunk Einstein and that you could say "gravity is a pseudo-force." You wouldn't be alone in that assessment, however the majority of physicists still have a bit to say about that.

What is it that is causing the curvature of space-time? The abstruse answer is the "stress-energy tensor." That object is as difficult to go about defining as it was for us to say when a curve was a line. So I will neglect the rigor in favor of metaphor. The stress-energy tensor in Einstein's theory of gravity is the analogous entity that mass was in Newton's theory. There is a "physical dialogue" going on between space-time and matter. The stress-energy tensor is that line of communication. Space-time tells matter where to go, and matter tells space-time how to curve. This dialogue is the essence of Einstein's theory.

The curvature of space-time distorts the manifold of the universe from a simply Euclidean space. The intrinsic idea of our covariant derivative is operationally defined as local reference frames. The local frames are ways of translating how we look at space-time, one can think of them as different coordinate charts. For any gravitational field, there exists a local inertial frame that is accelerating in the opposite direction of the gravitational field. The crazy bit is that operationally from physical measurements, we can't tell the difference between a gravitational field and an accelerated local frame. The way one can measure the force of gravity is by taking a test mass in a different neighborhood and measuring the distance from the mass as a function of time. The mass, unless acted on by a force, will move in a "straight" line through space-time. We can use this operation to measure the curvature of space, or if you are a physicist the magnitude of the gravitational field.

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