A TOPOLOGICAL QUANTUM FIELD THEORY OF INTERSECTION NUMBERS FOR MODULI SPACES OF ADMISSIBLE COVERS

by

Renzo Cavalieri

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

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THE UNIVERSITY OF UTAH GRADUATE SCHOOL

SUPERVISORY COMMITTEE APPROVAL

of a dissertation submitted by

Renzo Cavalieri

This dissertation has been read by each member of the following supervisory committee and by majority vote has been found to be satisfactory.

Chair: Aaron Bertram

Mladen Bestvina

Alastair Craw

Yuan Pin Lee

Dragan Milicic

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FINAL READING APPROVAL

To the Graduate Council of the University of Utah:

I have read the dissertation of Renzo Cavalieri in its final form and have found that (1) its format, citations, and bibliographic style are consistent and acceptable; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the Supervisory Committee and is ready for submission to The Graduate School.

Date

Aaron Bertram
Chair, Supervisory Committee

Approved for the Major Department

Graeme Milton
Chair/Dean

Approved for the Graduate Council

David S. Chapman
Dean of The Graduate School

ABSTRACT

This dissertation studies the intersection theory of moduli spaces of admissible covers. Following a parallel work in Gromov-Witten theory by Jim Bryan and Rahul Pandharipande, we define a natural class of intersection numbers on moduli spaces of admissible covers. We show that they can be organized in the structure of a two-dimensional, two-level weighted Topological Quantum Field Theory. Using techniques of localization, we compute the theory in low degrees, and provide a conjecture for general degree d. We then study two interesting specializations of the theory, where we are able to produce closed formulas for our invariants. These formulas involve characters from the representation theory of the symmetric group S_n, thus opening an interesting perspective for further exploration of the connection between the two theories.

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PREFACE

Gromov-Witten theory is the result of the merging of two apparently very different areas of science: on the one hand, the classical mathematical field of enumerative geometry; on the other, the young and tumultuous physical theories of strings and of mirror symmetry.

Enumerative geometry is an ancient branch of mathematics, whose purpose is to answer a natural class of questions:

"How many geometric objects of a certain type satisfy a given number of geometric conditions?"

We all know from grade school that there is a unique line through two points in the plane, and a unique conic through five points in general position in the plane. However, the going is steep, and already to determine that there are 12 rational planar cubics through eight points in general position is highly nontrivial.

A fair number of interesting results have been classically obtained, generally using ad hoc techniques and clever constructions. For example, there are exactly two lines incident to four general lines in three-dimensional space. To see this, one can degenerate the arrangement of four lines so that they are pairwise incident. It is then immediate to see the two desired lines: one goes through the points of intersection of the pairs of lines, the other is the intersection of the two planes supporting the pairs of lines. Finally, Schubert's Principle of Conservation of Number was generally accepted to believe that this is the general result.

Rising dissatisfaction in the field was given voice by David Hilbert, who asks in his 15th problem for solid mathematical foundations and a systematic approach to enumerative geometry.

A fundamental breakthrough in this direction was to develop a theory of moduli spaces, i.e., spaces parametrizing the geometric objects to be studied. Imposing geometric conditions corresponds to cutting appropriate subspaces in the moduli space. Thus enumerative geometry is reduced to intersection theory on moduli spaces.

To have an effective intersection theory on a space X, we need fundamentally three ingredients:

1. X should be compact.

2. X should not be horribly singular.

3. we should have a working understanding of the cohomology ring of X.

In general it is not possible to obtain all ingredients at the same time. Often moduli spaces that parametrize the objects we intend to study are not compact, as they do not naturally include the possible degenerations of such objects. The task of compactifying moduli spaces in a natural and operative way is then extremely important.

At present there is a certain understanding of moduli spaces of curves, maps from curves and between curves. In the case of surfaces, the situation is already extremely complicated, and very little is known.

In the early '90s, moduli spaces of curves became very important characters in theoretical physics. In the developement of a theory of quantum gravity, the count of holomorphic curves in the target space of the theory came to represent the quantum corrections in string compactification.

Further, with the birth of mirror symmetry in [CdlOGP91], the mysterious count of holomorphic curves could be conjecturally equated to a classical (and feasible) computation on the mirror manifold. Physicists have then been able to predict many amazing results in enumerative geometry, such as the celebrated prediction for the number of rational curves in the quintic threefold in $\mathbb{P}^4$.

The intertwining of physics and mathematics has continued to grow since. For physicists, verifying mathematically their prediction is an important and nontrivial check of the consistency of string theory; for mathematicians, physics is providing a wealth of conjectures, new ideas and methods to attack extremely natural and interesting mathematical problems.

A History of the Problem

In [BP03], Jim Bryan and Rahul Pandharipande study the Gromov-Witten invariants of curves embedded in an open Calabi-Yau threefold, and show that they can be organized to form a two-dimensional Topological Quantum Field Theory (TQFT). This fundamentally means that we can obtain all invariants combinatorially from a very limited number of generators for the theory. The Frobenius algebra associated to the TQFT is shown to be semisimple, meaning that this set of generators can be collapsed to just one, provided that we understand the semisimple basis for the Hilbert space of the theory. Unfortunately, this semisimple basis is quite mysterious ans has resisted many attempts at being described so far.

A few years later, [BP04] inserts the previous structure in a much broader and computationally convenient framework. The Calabi-Yau TQFT is seen as a specialization of a two-level weighted TQFT encoding the equivariant Gromov-Witten theory of curves embedded in open threefolds as the zero section of a rank two vector bundle. This has allowed one to compute the theory explicitely in low degrees, and produce topological recursion relations to compute it, in principle, for arbitrary degree.

This dissertation develops a parallel theory for intersection numbers on moduli spaces of admissible covers. These spaces are an alternative compactification of the moduli space of maps from smooth curves to a target curve, that have some advantages over the classical moduli spaces of stable maps:

- moduli spaces of admissible covers are smooth stacks;
- an admissible cover keeps track of the degree of the map on every irreducible component of the source curve, thus seeming a more appropriate vehicle for enumerative information.

We are able to set up a two-level weighted TQFT, and compute explicitly the theory in low degrees via localization. Our localization computations involve topological recursions between Hodge-type integrals on spaces of admissible covers.

Hodge integrals are a class of intersection numbers on moduli spaces of curves involving the tautological classes λ_i, which are the Chern classes of the Hodge bundle $\mathbb{E}$. In recent years Hodge integrals have shown a great amount of interconnections with Gromov-Witten theory and enumerative geometry.

The classical Hurwitz numbers, counting the numbers of ramified covers of a curve with an assigned set of ramification data, can be computed via Hodge integrals. Simple Hurwitz numbers have been discussed in [ELSV99], [ELSV01] and [GV03]; progress towards double Hurwitz numbers has been made in [GJV03].

Various spectacular computations of Hodge integrals were carried out in the late 90s by Faber and Pandharipande ([FP00]). Their results have been used to determine the multiple cover contributions in the GW invariants of $\mathbb{P}^1$, thus extending the well-known Aspinwall-Morrison formula in Gromov-Witten Theory.

Hodge integrals are also at the heart of the theory developed in [BP04], studying the local Gromov-Witten theory of curves.

Outline of the Dissertation

Chapter 1 gives an introduction to moduli spaces of admissible covers. The basic ideas and constructions are outlined, and the many variations on the main theme are shortly presented. We also introduce the tautological classes that are used in the following computations.

Chapter 2 is a quick guide to the technique of localization. This is a classical technique, and it is considered standard by the experts; however, it seemed useful to show with some detail how it works. We explicitely compute the restriction to the fixed loci of the bundles we are interested in, and the Euler class of the normal bundle to the fixed loci.

Chapter 3 introduces topological quantum field theories from an algebraic geometry point of view. The main facts and structure theorems are stated, and a definition of a weighted TQFT is given.

Chapter 4 constitutes the core of this work. The admissible cover invariants are defined and organized in TQFT structure. The main building blocks for the TQFT are computed via localization in degrees 1,2 and 3, and a conjecture is presented for the general case. Two specializations of the equivariant theory are studied.

For the anti-diagonal action of a one-dimensional torus, explicit closed formulas for the structure of the TQFT are given in terms of characters of the Representation Theory of the Symmetric group S_d. The TQFT is shown to be a one parameter deformation of the classical TQFT of Hurwitz numbers studied by Dijkgraaf and Witten in [DW90].

For the diagonal action, an explicit diagonalization of the theory is given in degree 2. For degree 3, we are able to prove an integrality result for the closed invariants.

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CHAPTER 1

ADMISSIBLE COVERS

Moduli spaces of admissible covers are a "natural" compactification of the Hurwitz schemes, parametrizing ramified covers of smooth Riemann Surfaces. The fundamental idea is that, in order to understand limit covers, we allow the base curve to degenerate together with the cover. Branch points are not allowed to "come together"; as two or more branch points tend to collide, a new component of the base curve sprouts from the point of collision, and the points transfer onto it. Similarly, upstairs the cover splits into a nodal cover.

Now more formally: let $(X, p_1, \cdots, p_n)$ be an n-pointed nodal curve of genus g.

Definition 1 *An **admissible cover** $\pi : E \longrightarrow X$ of degree d is a finite morphism satisfying the following:*

1. *E is a nodal curve.*
2. *Every node of E maps to a node of X.*
3. *The restriction of $\pi : E \longrightarrow X$ to $X \setminus (p_1, \cdots, p_n)$ is étale of constant degree d.*
4. *Over a node, locally in analytic coordinates, X, E and π are described as follows, for some positive integer r not larger than d:*

$$\begin{array}{rl} E: & e_1 e_2 = a, \\ X: & x_1 x_2 = a^r, \\ \pi: & x_1 = e_1^r, \ x_2 = e_2^r. \end{array}$$

Moduli spaces of admissible covers were introduced originally by Harris and Mumford in [HM82]. Intersection Theory on these spaces was for a long time extremely hard and mysterious, mostly because they are in general not normal, even if the normalization is always smooth. Only recently in [ACV01], Abramovich, Corti and Vistoli exhibit this normalization as the stack of balanced stable maps of degree 0 from twisted curves to the classifying stack $\mathcal{B}S_d$. This way they attain both the smoothness of the stack and a nice moduli-theoretic interpretation of it. We will abuse notation and refer to the Abramovich-Corti-Vistoli (ACV) spaces as admissible covers.

About at the same time, Ionel developed a parallel theory in the symplectic category ([Ion02]). Her moduli spaces are a generalization of the ACV ones, as will be discussed in section 1.1.5; the extra structure allows her to effectively do intersection theory on these spaces.

An introductory account of admissible covers of the sphere can be found in [HM98].

1.1 The Network of Admissible Covers Spaces

As soon as one starts paying close attention to moduli spaces of admissible covers, it becomes evident that there are many variations on the main theme. One can talk about admissible covers of a given base curve, or let the base curve be free to vary in moduli; one can allow the branch points to move around your base or require them to be fixed. One can mark or not mark points on the base and on the cover itself. One can restrict one's attention to connected covers or allow disconnected curves as well.

The corresponding spaces are strictly related one to the other. There are canonical maps connecting them into a rich network of geometric objetcs, and allowing us to choose the most appropriate setting for the problem at hand. In this section we want to present in a "working" way the basic ideas organizing this rich structure.

Definition 2 *Fix $d \geq 1$, and let $\mu_1, \cdots, \mu_n$ be partitions of d. We denote by*

$$\overline{Adm}_{h \xrightarrow{d} g, (\mu_1, \cdots, \mu_n)}$$

the disjoint union of connected components of the stack of balanced stable maps of degree 0 from a genus g, n-pointed twisted curve to $\mathcal{B}S_d$ characterized by the following conditions on the geometric points:

1. *the associated admissible cover (according to the construction in [ACV01], pag.3566) is a nodal curve of genus h.*
2. *let $x_1, \cdots, x_n$ be the marks on the base curve; the ramification profile over x_i is required to be of type μ_i.*

We call this the stack of admissible covers of degree d and genus h of a genus g curve.

If the equation

$$2h - 2 + d(2 - 2g) + \sum \ell(\mu_i) = nd \tag{1.1}$$

is not satisfied, these conditions define an empty space. If it is satisfied, we have a smooth stack of dimension $3g - 3 + n$. It admits two natural maps into moduli spaces of curves, as represented in the following diagram:

$$\begin{array}{ccc} \overline{Adm}_{h \xrightarrow{d} g,(\mu_1,\cdots,\mu_n)} & \rightarrow & \overline{M}_h \\ \downarrow & & \\ \overline{M}_{g,n} & & \end{array}$$

The map to $\overline{M}_h$ just looks at the source curve forgetting the cover map. The vertical morphism looks instead at the target curve, and at the (ordered) branch points. In particular, the vertical map has finite fibers.

1.1.1 Connected Admissible Covers

It is possible to consider moduli spaces parametrizing covers whose source curve is required to be connected. It is easy to express a moduli space of (possibly disconnected) admissible covers combinatorially in terms of finite disjoint unions, fiber products and quotients of connected admissible covers of possibly lower source genus and degree. For most purposes, the theory of admissible covers is more naturally set in the possibly disconnected domain. However, when explicitly doing intersection theory via localization, it is often easier to run the computations on the moduli spaces of connected covers, and then obtain the result in the disconnected case by exponentiation.

As a notation, we will "bullet" the spaces of connected admissible covers.

Definition 3 *Fix $d \geq 1$, and let $\mu_1, \cdots, \mu_n$ be partitions of d. We denote by*

$$\overline{Adm}^{\bullet}_{h \xrightarrow{d} g,(\mu_1,\cdots,\mu_n)}$$

the space of degree d admissible covers of a genus g curve by a connected curve of genus h, with ramification conditions specified by the partitions $\mu_1, \ldots, \mu_n$.

1.1.2 Symmetry

Consider the moduli space of admissible covers

$$\overline{Adm}_{h \xrightarrow{d} g,(\eta,\ldots,\eta,\mu_1,\cdots,\mu_n)},$$

where the first k ramification conditions are all equal to η.

Then the symmetric group on k letters S_k acts naturally on the moduli space, and we can consider the stack quotient

$$[\overline{Adm}_{h\xrightarrow{d}g,(\eta,\ldots,\eta,\mu_1,\cdots,\mu_n)}/S_k].$$

There is a natural étale quotient map of degree $k!$ between the two spaces.

We want to think of this quotient as parametrizing covers that have a given ramification data, but some of the branch points corresponding to ramification η are not marked.

Abuse of notation: simple ramification has a special role in the theory of covers. It constitutes, in some sense, the elementary building blocks for any kind of ramification, just how simple transpositions are building blocks for any permutation in the symmetric group. We use this idea to give a simple notation to a class of admissible cover spaces that will be essential for our purposes.

Definition 4 *Fix $d \geq 1$, and let $\mu_1, \cdots, \mu_n$ be partitions of d. Let g and h be integers. Suppose there is a positive integer k such that*

$$2h - 2 + d(2-2g) + \sum \ell(\mu_i) - k = nd.$$

Abusing notation, we denote by

$$\overline{Adm}_{h\xrightarrow{d}g,(\mu_1,\cdots,\mu_n)}$$

the stack quotient

$$[\overline{Adm}_{h\xrightarrow{d}g,(t_1,\ldots,t_k,\mu_1,\cdots,\mu_n)}/S_k],$$

where t stands for "transposition" and denotes the partition $(2,1,\ldots,1)$ associated to simple ramification.
We call this the space of degree d admissible covers of a genus g curve by a genus h curve, with specified ramification $\mu_1,\ldots,\mu_n$.

This is a smooth stack of dimension $3g-3+n+k$ and it admits the usual natural maps:

$$\begin{array}{ccc} \overline{Adm}_{h\xrightarrow{d}g,(\mu_1,\cdots,\mu_n)} & \to & \overline{M}_h \\ \downarrow & & \\ \overline{M}_{g,n} & & \end{array}$$

Notice, however, that now the vertical morphism has no longer finite fibers, but is of relative dimension k.

1.1.3 Admissible Covers of a Curve X

There is a natural way to construct the moduli space of admissible covers of a fixed curve X. Consider the map from a point into the moduli space $\overline{M}_g$ having image the geometric point $[X]$.

Definition 5 *The moduli space of degree d, genus h, admissible covers of a fixed curve X with prescribed ramification $\mu_1, \ldots, \mu_n$ is defined to be the following fiber product:*

$$\begin{array}{ccc} \overline{Adm}_{h \xrightarrow{d} X,(\mu_1,\cdots,\mu_n)} & \rightarrow & \overline{Adm}_{h \xrightarrow{d} g,(\mu_1,\cdots,\mu_n)} \\ \downarrow & & \downarrow \\ pt & \xrightarrow{[X]} & \overline{M}_g \end{array}$$

This is a smooth stack of dimension $n+k$, where k is defined as in the previous paragraph. If X is a nodal curve, then we are able to relate the moduli space of admissible covers of X to moduli spaces of admissible covers of the irreducible components of X. We will make these concepts precise in section 1.3.1.

1.1.4 Fixing Points

A similar procedure allows us to fix some of the ramification points on the base curve. Just for simplicity, let us fix one point at a time. Let (X, x_1) be a one pointed stable curve of genus g. Then the fiber product

$$\begin{array}{ccc} \overline{Adm}_{h \xrightarrow{d} X,(\mu_1 x_1,\cdots,\mu_n)} & \rightarrow & \overline{Adm}_{h \xrightarrow{d} g,(\mu_1,\cdots,\mu_n)} \\ \downarrow & & \downarrow \\ pt & \xrightarrow{[(X,x_1)]} & \overline{M}_{g,1} \end{array}$$

defines the space of admissible covers of X where the first ramification point is forced to lie over x_1. This is also a smooth stack and the dimension is $n + k - 1$. By repeating this procedure with other (distinct) points of X we can define spaces with an arbitrary number of fixed specified ramification.

Caveat: fixing multiple coincident points. Suppose we want to force more than one branch point to lie over a given point x. Then there is some subtlety to be dealt with. Our previous procedure can still be applied, where the coincident points transfer onto a sprouted genus 0 twig. In that case, though, the points are not fixed on the curve X, but on the "stable companion" $X \cup_x \mathbb{P}^1$. It is possible to define

$$\overline{Adm}_{h \xrightarrow{d} X, (\mu_1 x, \cdots, \mu_n x)} := \bigcup_{p_i \in \mathbb{P}^1} \overline{Adm}_{h \xrightarrow{d} X \cup_x \mathbb{P}^1, (\mu_1 p_1, \cdots, \mu_n p_n)}.$$

However, these spaces are in general singular.

1.1.5 Marking Points Upstairs

In [Ion02], Ionel constructs spaces of admissible covers where one marks some of the ramification points on the cover as well as the corresponding branch points. Trying to give a completely general notation becomes extremely burdensome. To get the concept, let us talk about spaces with only one specified branch point. Let $\mu = (\mu^1, \ldots, \mu^r)$ be one partition of the integer d, and consider the space:

$$\overline{Adm}_{h \xrightarrow{d} g, (\mu)}.$$

This space parametrizes covers that have a branch point y with ramification profile μ over it. The preimage of y consists of r distinct points. Let $0 \leq s \leq r$. Consider a subpartition of μ of length s. If we allow ourselves to reorder the elements in the partition, neglecting the strict ordering condition, we can assume that the subpartition consists of the first s elements in μ. We define the enriched partition $\hat{\mu}$ to simply be a labeling of the s parts in the subpartitions with the symbols p_i' :

$$\hat{\mu} := (\mu^1 \star p_1, \ldots, \mu^s \star p_s, \mu^{s+1}, \ldots, \mu^r).$$

By

$$\overline{Adm}_{h \xrightarrow{d} g, (\hat{\mu})}$$

we denote the moduli space of degree d admissible covers of a genus g curve by a genus h curve, with one specified branch point y having ramification profile μ, and s points marked on the preimage of y. The point p_i marks a point around which the local expression of the cover is $z \mapsto z^{\mu_i}$.

There is a natural forgetful map

$$\overline{Adm}_{h \xrightarrow{d} g, (\hat{\mu})} \longrightarrow \overline{Adm}_{h \xrightarrow{d} g, (\mu)}.$$

The degree of such a map is a simple combinatorial computation. Rather than trying to write a general formula that would be completely unreadable, let us just explain how to compute it. Suppose our ramification condition μ contains m_1 parts of type μ_1, and that we choose to mark n_1 of them with p_i's. Then the μ_1's contribute a factor of

$$\binom{m_1}{n_1} n_1!$$

to the degree of the map. In particular, if we choose to mark all the ramification upstairs, and we rewrite the partition μ as $((\mu^1)^{m_1} \dots (\mu^t)^{m_t})$, then the degree of the forgetful map is simply $\prod m_i!$.

Notation: from now on, unless otherwise specified, we will denote by $\hat{\eta}$ the enriched partition corresponding to marking all the preimages of the branch point corresponding to η.

1.1.6 The Genus 0 Case

Moduli spaces of admissible covers of $\mathbb{P}^1$ are at the same time the simplest and the most delicate, because genus 0 curves do not have moduli, but have a three-dimensional group of automorphisms. For this reason, and because these are the spaces on which we will be running explicit computations, we do give an independent treatment of this case. There are two possible incarnations of these spaces, according to whether we want to think of having fixed a particular parametrization of $\mathbb{P}^1$ or not.

1.1.6.1 Admissible Covers of an Unparametrized $\mathbb{P}^1$

Definition 6 *Fix $d \geq 1$, and let $\mu_1, \cdots, \mu_n$ be partitions of d. Let h be an integer, and assume that the condition*

$$2h - 2 + 2d + \sum \ell(\mu_i) = nd \tag{1.2}$$

holds. We denote by

$$\overline{Adm}_{h \xrightarrow{d} 0, (\mu_1, \cdots, \mu_n)}$$

the union of connected components of the stack of balanced stable maps of degree 0 from a genus 0, n-pointed twisted curve to $\mathcal{B}S_d$ characterized by the following conditions:

1. *the associated admissible cover (according to the construction in [ACV01], pag.3566) is a nodal curve of genus h.*

2. *let $x_1, \cdots, x_n$ be the marks on the base curve; the ramification profile over x_i is required to be of type μ_i.*

We call this the stack of admissible covers of degree d and genus h of a genus 0 curve.

This is a smooth stack of dimension $n-3 = 2h+2d+n+\sum \ell(\mu_i)-nd-5$, where $\ell(\mu_i)$ denotes the length of the partition μ_i. It admits two natural maps into moduli spaces of curves, as represented in the following diagram:

$$\begin{array}{ccc} \overline{Adm}_{h \xrightarrow{d} 0,(\mu_1,\cdots,\mu_n)} & \rightarrow & \overline{M}_h \\ \downarrow & & \\ \overline{M}_{0,n} & & \end{array}$$

In particular, the vertical map has finite fibers.

As before, if condition (1.2) is not satisfied, we add the appropriate number of simple transpositions and then consider the stack quotient by the action of the symmetric group on the added transpositions.

1.1.6.2 Admissible Covers of a Parametrized $\mathbb{P}^1$

The objects we parametrize are the same as above, but the equivalence relation is stricter: we consider two covers $E_1 \to \mathbb{P}^1$, $E_2 \to \mathbb{P}^1$ equivalent if there is an isomorphism $\varphi : E_1 \to E_2$ that makes the natural triangle commute. In other words, we are not allowed to act on the base with an automorphism of $\mathbb{P}^1$.

Definition 7 *We denote by*

$$\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,(\mu_1,\cdots,\mu_n)}$$

the stack of admissible covers of degree d of (a parametrized) $\mathbb{P}^1$ by curves of genus h, with n specified branch points having ramification profile $\mu_1, \cdots, \mu_n$.

We construct the space of parametrized admissible covers as the stack of balanced stable maps of degree $d!$ from the category of genus 0, n-pointed twisted curves to the stack quotient $[\mathbb{P}^1/S_d]$, where S_d acts trivially on $\mathbb{P}^1$. This is but a slight variation to the ACV construction. We illustrate what happens over a geometric point Spec($\mathbb{C}$)in Figure 1.1.

A map of degree $d!$ from the twisted curve produces a map of degree 1 from the coarse curve (and this is our desired parametrization of one special genus 0 twig on the base), a principal S_d bundle over the twisted curve and an S_d equivariant map to $\mathbb{P}^1$ (this data characterizes the admissible cover). Two admissible covers are equivalent if there is an automorphism of the twisted curve that makes them commmute. In doing so, the degree 1 map to $\mathbb{P}^1$ has to be respected, so only the nonparametrized twigs are free to be acted upon by automorphisms. This is illustrated in Figure 1.2.

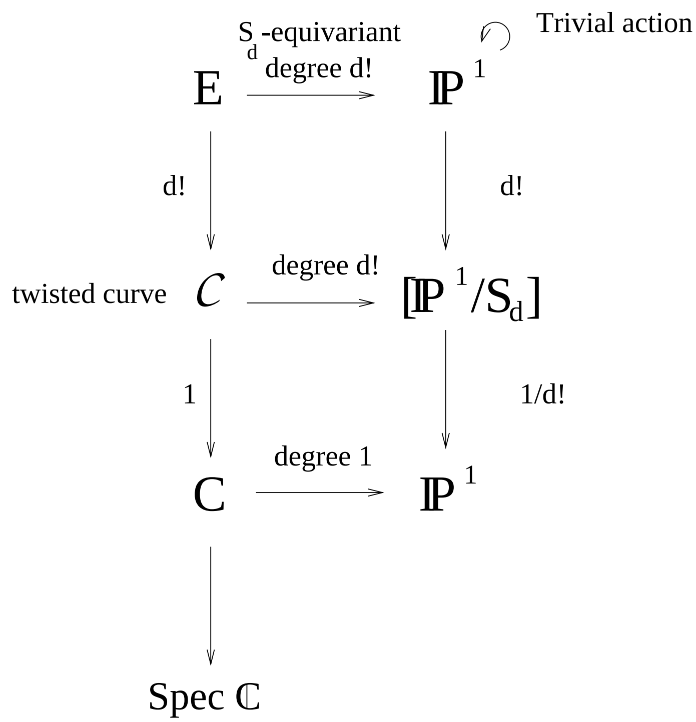


Figure 1.1. The stack of admissible covers of a parametrized $\mathbb{P}^1$.

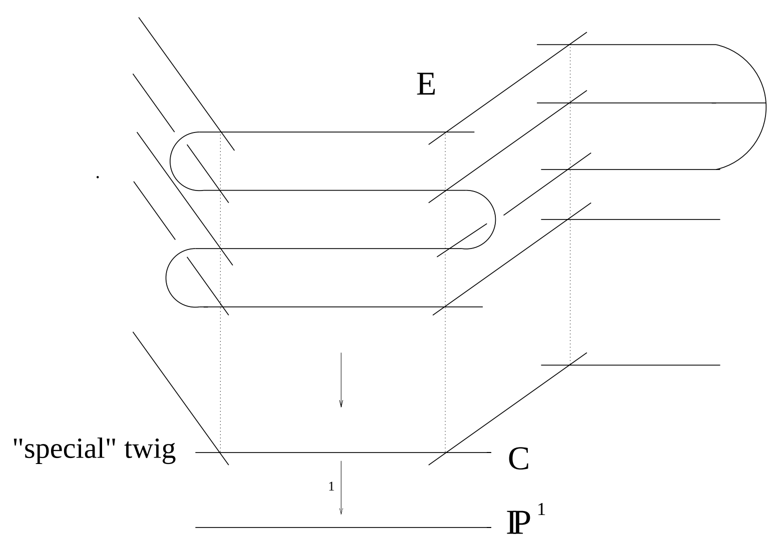


Figure 1.2. Schematic depiction of an admissible cover of a parametrized $\mathbb{P}^1$.

This is either empty or a smooth stack of dimension $n = 2h+2d+n+\sum \ell(\mu_i)-nd-2$, admitting two natural morphisms

$$\begin{array}{ccc} \overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,(\mu_1,\cdots,\mu_n)} & \rightarrow & \overline{M}_h \\ \downarrow & & \\ \mathbb{P}^1[n] & & \end{array}$$

The map to $\overline{M}_h$ just looks at the source curve forgetting the cover map. The vertical morphism, taking values in the Fulton-Mac Pherson configuration space of n points in $\mathbb{P}^1$, looks instead at the target curve, and at the (ordered) branch points.

1.2 Universal Families

The stacks of admissible covers admit a universal base family $\mathcal{U}$, a universal cover family $\mathcal{V}$ and a universal cover map φ. The cover map takes values in a stack $\mathcal{X}$, that is a family over the moduli space. The fiber $\mathcal{X}_\bullet$ over a moduli point consists of a nodal, genus g curve, possibly with rational bubbles attached at nodes. The universal cover map can be followed by a map ε, that contracts all secondary twigs and takes values in $\overline{Adm} \times \overline{M}_{g,n+1}$, where we want to think of the stack $\overline{M}_{g,n+1}$ as the universal family over $\overline{M}_{g,n}$. The situation is illustrated by the following diagram:

$$\begin{array}{ccccccccc} & \mathcal{V} & & & & & & & \\ p & \downarrow & \searrow & & & & & & \\ & \mathcal{U} & \xrightarrow{\varphi} & \mathcal{X} & \xrightarrow{\varepsilon} & \overline{Adm}_{h \xrightarrow{d} g,(\mu_1,\cdots,\mu_n)} \times \overline{M}_{g,n+1} & \rightarrow & \overline{M}_{g,n+1} \\ \pi & \downarrow & \swarrow & & & & & & \\ \overline{Adm}_{h \xrightarrow{d} g,(\mu_1,\cdots,\mu_n)} & & & & & & & & \end{array}$$

We call f the composition of the three horizontal maps.

1.2.1 The Universal Base Family

The universal base family can itself be interpreted as a moduli space of admissible covers. If we think of admissible covers as of stable maps from a twisted curve, then we obtain a universal family by adding a mark to the twisted curve and requiring trivial ramification over it. Let us denote by (1^d) the partition $(1,\ldots,1)$ of d, representing an unramified point. Then,

$$\mathcal{U} = \overline{Adm}_{h\xrightarrow{d}g,(\mu_1,\cdots,\mu_n,(1^d))}.$$

We define n tautological sections

$$\sigma_i : \overline{Adm}_{h\xrightarrow{d}g,(\mu_1,\cdots,\mu_n)} \longrightarrow \overline{Adm}_{h\xrightarrow{d}g,(\mu_1,\cdots,\mu_n,(1^d))}$$

of the natural forgetful map. The image of the i-th section consists of covers where a new rational component has sprouted from the i-th marked point. The marked points (1^d) and μ_i have transferred onto this twig. Over this twig we find $\ell(\mu_i)$ copies of $\mathbb{P}^1$ fully ramified over the attaching point and over the marked point μ_i, as shown in Figure 1.3.

Finally we can define the natural evaluation maps:

$$ev_i := f \circ \sigma_i : \overline{Adm}_{h\xrightarrow{d}g,(\mu_1,\cdots,\mu_n)} \longrightarrow \overline{M}_{g,n+1}.$$

1.2.2 The Universal Cover Family

The universal cover family also admits an interpretation as a moduli space of admissible covers. We need to appeal to the moduli spaces constructed by Ionel.

Let $(\hat{1^d})$ denote the enriched partition

$$(1 \star p_1, 1, \ldots, 1).$$

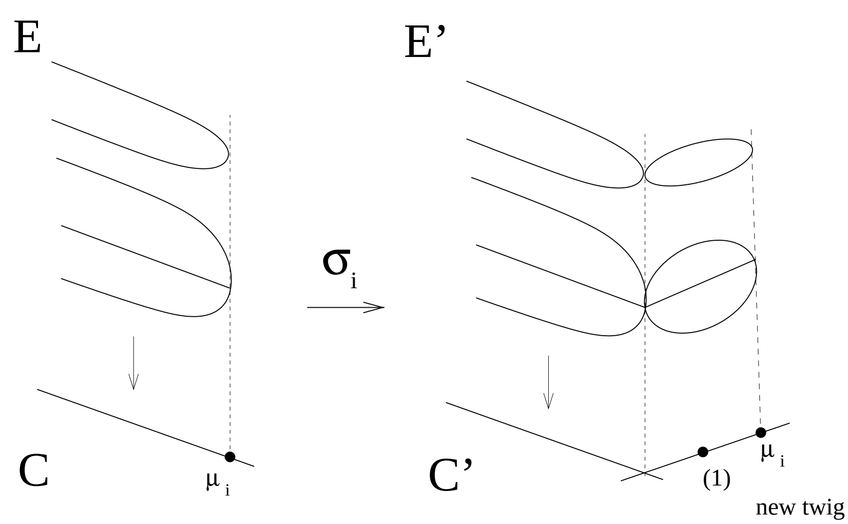


Figure 1.3. The tautological section σ_i.

We are choosing to mark only one point on the unramified preimage of the marked point downstairs. Then the universal family $\mathcal{V}$ is precisely the moduli space

$$\overline{Adm}_{g\xrightarrow{d}\mathbb{P}^1,(\mu_1,\cdots,\mu_n,(\hat{1^d}))}.$$

The map $p : \mathcal{V} \to \mathcal{U}$ is the natural degree d forgetful map described in section 1.1.5.

1.3 The Boundary

The boundary of spaces of admissible covers can be described in terms of admissible cover spaces of possibly lower degree or genus. In the case of admissible covers of a parametrized $\mathbb{P}^1$, the boundary involves also admissible covers of an unparametrized genus 0 curve. In Figure 1.2, for example, we can obtain the depicted admissible cover by "gluing together" one admissible cover of a parametrized $\mathbb{P}^1$ (the cover of the special twig) and three admissible covers of an irreducible genus 0 curve.

It would be very tempting to conclude that the irreducible boundary components of an admissible cover space are actually products of other admissible cover spaces; however we need to be very careful, and consider the contribution to the stack structure given by automorphisms.

To illustrate this point let us carefully analyze the gluing map. For simplicity of exposition, let us first glue at a fully ramified point. Denote by $\mathcal{B}$ the irreducible component of the boundary parametrizing covers such that:

- the base curve is a nodal curve whose irreducible components are of genus g_1 and g_2;
- the cover is a nodal curve whose irreducible components are of genus h_1 and h_2;
- the profile over the node is described by the partition (d).

Then a gluing map is defined:

$$\begin{array}{ccl} \overline{Adm}_{h_1\xrightarrow{d}g_1,(\mu_1,\cdots,\mu_{n_1},(d))} & \times & \overline{Adm}_{h_2\xrightarrow{d}g_2,(\lambda_1,\cdots,\lambda_{n_2},(d))} \\ & \downarrow & \\ & \mathcal{B} & \hookrightarrow \overline{Adm}_{h_1+h_2\xrightarrow{d}g_1+g_2,(\mu_1,\cdots,\mu_{n_1},\lambda_1,\cdots,\lambda_{n_2})} \end{array}$$

We claim that the vertical map is an étale map of stacks of degree $1/d$. Let us look at a point $[E \to X]$ of $\mathcal{B}$: we observe that it admits a unique preimage $([E_1 \to X_1], [E_2 \to X_2])$, and we count the automorphisms of the preimage modulo automorphisms pulled-back from below. In local analytic coordinates around the node, the cover is described as

$$\begin{array}{c} Spec(\mathbb{C}[e_1,e_2]/(e_1e_2-a)) \\ \downarrow \\ Spec(\mathbb{C}[x_1,x_2]/(x_1x_2-a^d)), \end{array}$$

by the local equations $x_1 = e_1^d, x_2 = e_2^d$. Modding out by automorphisms of the "glued" cover is equivalent to requiring the first coordinate e_1 to remain untouched. It is then evident that what we have left are d distinct automorphisms, consisting in multiplying e_2 by a d^{th} root of unity. This establishes our claim.

Now if we want to glue two branch points with ramification profile $\eta = (\eta^1, \ldots, \eta^k)$, the situation will be analogous. There is one subtlety that we need to deal with. If the d_i's are not all distinct, we need to be able to distinguish them in order for a gluing map to be defined. Again, the concept is simple but a general notation is cumbersome, so let us first develop the extreme case in which all of the $\eta^i = \alpha$ are the same. Let us consider the enriched partition

$$\hat{\eta} = (\alpha \star p_1, \ldots, \alpha \star p_k),$$

and the corresponding Ionel moduli spaces.

The gluing map

$$\begin{array}{ccc} \overline{Adm}_{h_1 \stackrel{d}{\to} g_1,(\mu_1,\cdots,\mu_{n_1},\hat{\eta})} & \times & \overline{Adm}_{h_2 \stackrel{d}{\to} g_2,(\lambda_1,\cdots,\lambda_{n_2},\hat{\eta})} \\ & \downarrow & \\ & \mathcal{B}' & \hookrightarrow \overline{Adm}_{h_1+h_2+k-1 \stackrel{d}{\to} g_1+g_2,(\mu_1,\cdots,\mu_{n_1},\lambda_1,\cdots,\lambda_{n_2})} \end{array} \tag{1.3}$$

is an étale map of stacks of degree $k!/\alpha^k$.

Finally, for a general partition

$$\eta = ((\eta^1)^{m_1}, \ldots, (\eta^k)^{m_k})$$

the gluing map (3.3) is an étale map of stacks of degree

$$\frac{\prod m_i!}{(\eta^1)^{m_1} \cdot \ldots \cdot (\eta^k)^{m_k}}.$$

These maps describe all irreducible components of the boundary parametrizing reducible base curves. A similar reasoning applies to irreducible nodal base curves. Given a partition η of d, and the enriched partition $\hat{\eta}$ as above, we can define a gluing map:

$$\begin{array}{ccc} \overline{Adm}_{h \stackrel{d}{\to} g,(\mu_1,\cdots,\mu_n,\hat{\eta},\hat{\eta})} & & \\ \downarrow & & \\ \mathcal{B}' & \hookrightarrow & \overline{Adm}_{h+\ell(\eta) \stackrel{d}{\to} g+1,(\mu_1,\cdots,\mu_n)} \end{array} \tag{1.4}$$

Just as before the vertical map is an étale map of stacks of degree

$$\frac{\prod m_i!}{(\eta^1)^{m_1} \cdot \ldots \cdot (\eta^k)^{m_k}}.$$

1.3.1 Admissible Covers of a Nodal Curve

The gluing maps discussed in section 1.3 allow us to describe completely moduli spaces of admissible covers of nodal curves.

1.3.1.1 Reducible Nodal Curves

Let

$$X = X_1 \bigcup_{x_1 = x_2} X_2$$

be a nodal curve of genus g, obtained by attaching at a point two irreducible curves of genus g_1 and g_2. We wish to describe the moduli space $\overline{Adm}_{h \xrightarrow{d} X}$.

For h_1, h_2 and η such that

$$h_1 + h_2 + \ell(\eta) - 1 = h, \tag{1.5}$$

denote by Φ_{η,h_1,h_2} the gluing map:

$$\Phi_{\eta,h_1,h_2} : \overline{Adm}_{h_1 \xrightarrow{d} X_1,(\hat{\eta})} \times \overline{Adm}_{h_2 \xrightarrow{d} X_2,(\hat{\eta})} \to \overline{Adm}_{h \xrightarrow{d} X}.$$

If we let η vary among all partitions of d, h_1 and h_2 vary among all pairs of integers satisfying condition (1.5), we obtain a collection of étale maps with disjoint images that cover the moduli space $\overline{Adm}_{h \xrightarrow{d} X}$.

As an element in the Chow ring with rational coefficients, we can then express:

$$[\overline{Adm}_{h \xrightarrow{d} X}] = \sum_{\eta,h_1,h_2} \frac{(\eta^1)^{m_1} \cdot \ldots \cdot (\eta^k)^{m_k}}{\prod m_i!} \Phi_{\eta,h_1,h_2*}\left([\overline{Adm}_{h_1 \xrightarrow{d} X_1,(\hat{\eta})}] \times [\overline{Adm}_{h_2 \xrightarrow{d} X_2,(\hat{\eta})}]\right).$$

For simplicity, we will omit the natural pushforward maps. Also, we want to express our result in terms of our ordinary spaces, not of the Ionel ones. To do so let us notice that $\overline{Adm}_{h_1 \xrightarrow{d} X_1,(\eta)}$ is an étale quotient of $\overline{Adm}_{h_1 \xrightarrow{d} X_1,(\hat{\eta})}$ of degree $\prod m_i!$.

Finally, if we define

$$\mathfrak{z}(\eta) := \prod m_i!(\eta^i)^{m_i},$$

we obtain the formula:

$$\boxed{[\overline{Adm}_{h\xrightarrow{d}X}] = \sum_{\eta,h_1,h_2} \mathfrak{z}(\eta)[\overline{Adm}_{h_1\xrightarrow{d}X_1,(\eta)}] \times [\overline{Adm}_{h_2\xrightarrow{d}X_2,(\eta)}].} \tag{1.6}$$

Now, if we are dealing with an admissible cover space with also a prescribed vector of ramification conditions $\underline{\mu}$, the previous reasoning continues to hold, with the only extra combinatorial complication of having to distribute the μ_i's on the two twigs X_1 and X_2 in all possible ways.

1.3.1.2 Irreducible Nodal Curves

Let

$$X = X'/\{x_1 = x_2\}$$

be a nodal curve of genus g, obtained by gluing two distinct points of an irreducible curve X' of genus $g-1$. We wish to describe the moduli space $\overline{Adm}_{h\xrightarrow{d}X}$.

For a given η, define the integer h' by the formula

$$h' + \ell(\eta) = h. \tag{1.7}$$

Then denote by Φ_η the gluing map:

$$\Phi_\eta : \overline{Adm}_{h'\xrightarrow{d}X',(\hat{\eta},\hat{\eta})} \to \overline{Adm}_{h\xrightarrow{d}X}.$$

If we let η vary among all partitions of d, we obtain a collection of étale maps with disjoint images that cover the moduli space $\overline{Adm}_{h\xrightarrow{d}X}$.
As an element in the Chow ring with rational coefficients, we can then express:

$$[\overline{Adm}_{h\xrightarrow{d}X}] = \sum_{\eta} \frac{(\eta^1)^{m_1} \cdot \ldots \cdot (\eta^k)^{m_k}}{\prod m_i!} [\overline{Adm}_{h'\xrightarrow{d}X',(\hat{\eta},\hat{\eta})}].$$

Again, going from Ionel to regular spaces:

$$\boxed{[\overline{Adm}_{h\xrightarrow{d}X}] = \sum_{\eta} \mathfrak{z}(\eta)[\overline{Adm}_{h'\xrightarrow{d}X',(\eta,\eta)}].} \tag{1.8}$$

1.4 Tautological Classes on Admissible Covers

We are interested in describing some "tautological" intersection classes on the stacks of admissible covers. We want to endow our spaces with analogues of λ and ψ classes. These classes can be defined on all the spaces of admissible covers we have described. However, since we will be explicitly using them in computations on admissible covers of an unparametrized genus 0 curve, we choose to develop the definitions only in this particular case. The modifications required for all other cases are straightforward. To define λ and ψ classes we will simply pull-back these classes from the appropriate moduli spaces.

Recall the forgetful map:

$$\overline{Adm}_{h \xrightarrow{d} 0,(\eta_1,\cdots,\eta_n)} \xrightarrow{s} \overline{M}_h.$$

Recall that the tautological class $\lambda_i \in A^i(\overline{M}_h)$ is defined to be the i-th Chern class of the Hodge bundle $\mathbb{E}$.

Definition 8 *The tautological class $\lambda_i^{Adm} \in A^i(\overline{Adm}_{h \xrightarrow{d} 0,(\eta_1,\cdots,\eta_n)})$ is defined to be the i-th Chern class of the pull-back of the Hodge bundle via the map s:*

$$\lambda_i^{Adm} := s^*(\lambda_i).$$

We will drop the superscript "Adm" and simply write λ_i whenever there is no risk of confusion.

Let us now look at another natural map:

$$\overline{Adm}_{h \xrightarrow{d} 0,(\eta_1,\cdots,\eta_n)} \xrightarrow{t} \overline{\mathcal{M}}_{0,n}.$$

The stack $\overline{\mathcal{M}}_{0,n}$ is the moduli space of twisted n-pointed curves of genus 0.
Let $\overline{\mathcal{M}}_{0,n+1} \xrightarrow{\pi} \overline{\mathcal{M}}_{0,n}$ be the universal family over this stack, $\omega_\pi \to \overline{\mathcal{M}}_{0,n+1}$ be the relative dualizing sheaf and σ_i the i-th tautological section. Then $\psi_i \in A^1(\overline{\mathcal{M}}_{0,n})$ is defined to be the first Chern class of $\sigma_i^*(\omega_\pi)$.

Definition 9 *The tautological class $\psi_i^{Adm} \in A^1(\overline{Adm}_{h \xrightarrow{d} 0,(\eta_1,\cdots,\eta_n)})$ is defined to be the pull-back of the analogous class via the map t:*

$$\psi_i^{Adm} := t^*(\psi_i).$$

Again, the superscript will be dropped unless needed for clarity.

We can also view ψ classes in a more intrinsic fashion. Consider:

- the space

$$\overline{Adm}_{h\xrightarrow{d}0,(\eta_1,\cdots,\eta_n,(1^d))}$$

where we have added a trivial ramification condition;

- the forgetful map

$$\pi_{(1^d)}:\overline{Adm}_{h\xrightarrow{d}0,(\eta_1,\cdots,\eta_n,(1^d))}\longrightarrow\overline{Adm}_{h\xrightarrow{d}0,(\eta_1,\cdots,\eta_n)};$$

- the i-th tautological section

$$\sigma_i:\overline{Adm}_{h\xrightarrow{d}0,(\eta_1,\cdots,\eta_n)}\longrightarrow\overline{Adm}_{h\xrightarrow{d}0,(\eta_1,\cdots,\eta_n,(1^d))}.$$

Lemma 1 *The class $-\psi_i\in A^1(\overline{Adm}_{h\xrightarrow{d}0,(\eta_1,\cdots,\eta_n)})$ is the first Chern class of the normal bundle to the image of the section σ_i.*

PROOF: Observe the following commutative diagram:

$$\begin{array}{ccc} \overline{Adm}_{h\xrightarrow{d}0,(\eta_1,\cdots,\eta_n,(1^d))} & \xrightarrow{\tilde{t}} & \overline{\mathcal{M}}_{0,n+1} \\ \tilde{\sigma}_i\uparrow\downarrow\pi_{(1^d)} & \square & \sigma_i\uparrow\downarrow\pi \\ \overline{Adm}_{h\xrightarrow{d}0,(\eta_1,\cdots,\eta_n)} & \xrightarrow{t} & \overline{\mathcal{M}}_{0,n} \end{array}$$

We know from [ACV01], pag 3561, that the maps t and $\tilde{t}$ are étale onto their image. Further, the diagram is cartesian. Now our lemma follows from the analogous statement on $\overline{\mathcal{M}}_{0,n}$:

$$-\psi_i^{Adm}=t^*(-\psi_i)=c_1(t^*\sigma_i^*N_{\sigma_i})=c_1(\tilde{\sigma}_i^*\tilde{t}^*N_{\sigma_i})=c_1(\tilde{\sigma}_i^*N_{\tilde{\sigma}_i}).$$

1.4.1 Ionel's Lemma

The ψ classes we have defined are the first Chern classes of the cotangent line bundle at a marked point on the base curve. In the spaces constructed by Ionel, however, it is possible to talk about the cotangent line bundle of the upstairs curve at a marked ramification point, and to define "upstairs-ψ" classes. As to be expected, the two types of classes are strictly related.

Lemma 2 (Ionel,[Ion02], 1.17) *Consider the space*

$$\overline{Adm}_{h\xrightarrow{d}g,\hat{\eta}}$$

Denote by x the marked point on the base curves, and by p_i the marked ramification point corresponding to the part η_i.

Let ψ_x *be the class defined in section 1.4, and let* ψ_{p_i} *be the class obtained by pulling back the corresponding class on* $\overline{M}_{h,\ell(\eta)}$*. Then*

$$\boxed{\psi_x = \eta^i \psi_{p_i}.}$$

CHAPTER 2

LOCALIZATION

The main tool we use for evaluating intersection numbers on moduli spaces of admissible covers is the Atiyah-Bott localization theorem ([AB84]). We begin this chapter by briefly recalling the theorem. Then, we develop the set-up in which we use it. We observe that moduli spaces of admissible covers of a parametrized $\mathbb{P}^1$ come equipped with a natural torus action, and hence lend themselves to the techniques of localization. We explicitly compute the Euler class of a bundle that will be fundamental in our calculations later on. We express the Euler class of the normal bundle to the irreducible components of the fixed loci in terms of tautological classes on the moduli spaces.

2.1 Atiyah-Bott Localization Theorem

Consider the one-dimensional algebraic torus $\mathbb{C}^*$, and recall that the $\mathbb{C}^*$-equivariant Chow ring of a point is a polynomial ring in one variable:

$$A^*_{\mathbb{C}^*}(\{pt\}, \mathbb{C}) = \mathbb{C}[\hbar].$$

Let $\mathbb{C}^*$ act on a smooth, proper stack X, denote by $i_k : F_k \hookrightarrow X$ the irreducible components of the fixed locus for this action and by N_{F_k} their normal bundles. The natural map:

$$\begin{array}{ccc} A^*_{\mathbb{C}^*}(X) \otimes \mathbb{C}(\hbar) & \rightarrow & \sum_k A^*_{\mathbb{C}^*}(F_k) \otimes \mathbb{C}(\hbar) \\ \alpha & \mapsto & \dfrac{i_k^*\alpha}{c_{top}(N_{F_k})}. \end{array}$$

is an isomorphism. Pushing forward equivariantly to the class of a point, we obtain the Atiyah-Bott integration formula:

$$\int_{[X]} \alpha = \sum_k \int_{[F_k]} \frac{i_k^*\alpha}{c_{top}(N_{F_k})}.$$

2.2 Our Set-up

Let $\mathbb{C}^*$ act on a two-dimensional vector space V via:

$$t \cdot (z_0, z_1) = (tz_0, z_1).$$

This action descends on $\mathbb{P}^1$, with fixed points $0 = (1:0)$ and $\infty = (0:1)$. An equivariant lifting of $\mathbb{C}^*$ to a line bundle L over $\mathbb{P}^1$ is uniquely determined by its weights $\{L_0, L_\infty\}$ over the fixed points.

The canonical lifting of $\mathbb{C}^*$ to the tangent bundle of $\mathbb{P}^1$ has weights $\{1, -1\}$.

The action on $\mathbb{P}^1$ induces an action on the moduli spaces of admissible Covers to a parametrized $\mathbb{P}^1$ simply by postcomposing the cover map with the automorphism of $\mathbb{P}^1$ defined by t.

The fixed loci for the induced action on the moduli space consist of admissible covers such that anything "interesting" (ramification, nodes, marked points) happens over 0 or ∞, or on "nonspecial" twigs that attach to the main $\mathbb{P}^1$ at 0 or ∞.

2.3 Restricting Chow Classes to the Fixed Loci

We want to compute the restriction to various fixed loci of the top Chern class of the bundle

$$E = R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)).$$

The top Chern class $c_{2h+d-1}(E)$ splits as

$$c_{2h+d-1}(E) = c_h(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1})c_{h+d-1}(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)),$$

so we will analyze the two terms separately.

There is a standard technique to carry out these computations. To avoid an overwhelmingly cumbersome notation, we choose to show it only in a particular example, which will be the most important for our purposes.

Let us consider the fixed locus $F_{h_1 h_2}$, consisting of covers where the main $\mathbb{P}^1$ is ramified over 0 and ∞ and curves of genus h_1 and h_2 are attached on either side. A point in this fixed locus is represented in Figure 2.1, where we denote by X the nodal curve, C_1 and C_2 the irreducible components over 0 and ∞.

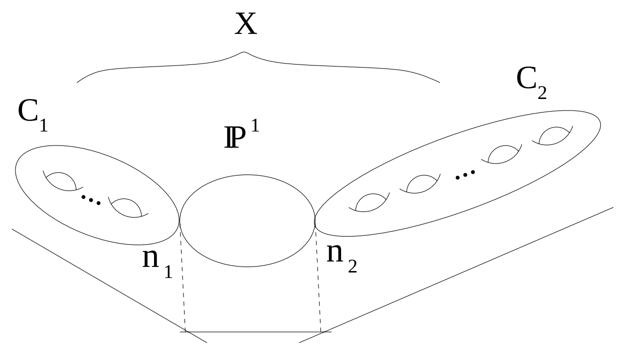


Figure 2.1. A cover in the fixed locus F_{h_1,h_2}.

The starting point in analyzing the restriction of the bundle E to this fixed locus is the classical *normalization sequence*:

$$0 \to \mathcal{O}_X \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{C_2} \to \mathbb{C}_{n_1} \oplus \mathbb{C}_{n_2} \to 0.$$

1) $c_h(R^1\pi_*f^*\mathcal{O}_{\mathbb{P}^1})$**:**

It suffices to analyze the long exact sequence in cohomology associated to the normalization sequence:

$$0 \to h^0(\mathcal{O}_X) \to h^0(\mathcal{O}_{C_1}) \oplus h^0(\mathcal{O}_{\mathbb{P}^1}) \oplus h^0(\mathcal{O}_{C_2}) \to \mathbb{C}_{n_1} \oplus \mathbb{C}_{n_2} \to$$

$$\to h^1(\mathcal{O}_X) \to h^1(\mathcal{O}_{C_1}) \oplus h^1(\mathcal{O}_{C_2}) \to 0.$$

Assume that $\mathcal{O}_{\mathbb{P}^1}$ is linearized with weights $\{\alpha, \alpha\}$. Then

$$c_h(R^1\pi_*f^*\mathcal{O}_{\mathbb{P}^1}) = (-1)^h \Lambda_{h_1}(-\alpha)\Lambda_{h_2}(-\alpha),$$

where the following notational convention holds:

$$\Lambda_h(n) = \sum (n\hbar)^i \lambda_{h-i}.$$

The reason for switching from α to $-\alpha$ is that $h^1(\mathcal{O})$ are the fibers of the dual bundle to the Hodge bundle, hence the odd degree Chern classes will have a negative sign.

2) $c_{h+d-1}(R^1\pi_*f^*\mathcal{O}_{\mathbb{P}^1}(-1))$**:**

In this case we first want to tensor the normalization sequence by $f^*\mathcal{O}_{\mathbb{P}^1}(-1)$, and then proceed to analyze the long exact sequence in cohomology:

$$0 \to h^0(\mathcal{O}_{C_1}) \oplus h^0(\mathcal{O}_{C_2}) \to \mathbb{C}_{n_1} \oplus \mathbb{C}_{n_2} \to$$

$$\to h^1(f^*\mathcal{O}_{\mathbb{P}^1}(-1)) \to h^1(\mathcal{O}_{C_1}) \oplus h^1(\mathcal{O}_{\mathbb{P}^1}(-d)) \oplus h^1(\mathcal{O}_{C_2}) \to 0.$$

Now, having linearized $\mathcal{O}_{\mathbb{P}^1}(-1)$ with weights $\{\beta, \beta+1\}$,

$$c_{h+d-1}(R^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)) = (-1)^h \Lambda_{h_1}(-\beta)\Lambda_{h_2}(-\beta-1)\hbar^{d-1} \prod_1^{d-1} \left(\beta + \frac{i}{d}\right).$$

The last term in our contribution, coming from $h^1(\mathcal{O}_{\mathbb{P}^1}(-d))$, is explained in the following way. Consider a degree d map from $\mathbb{P}^1$ to $\mathbb{P}^1$. The target curve is given the natural $\mathbb{C}^*$ action, and the tautological bundle is linearized with weights $\{\beta, \beta+1\}$. Now let x and z be local coordinates around 0 for, respectively, the target and the source curve. The expression of the map in local coordinates is

$$x = z^d.$$

We see then that z must have weight $-1/d$. The vector space $h^1(\mathcal{O}_{\mathbb{P}^1}(-d))$ is $(d-1)$ dimensional and generated, in local coordinates, by the sections $\{1/z, 1/z^2, \ldots, 1/z^{d-1}\}$. The vector bundle over moduli with these fibers is trivial, because $\mathbb{P}^1$ is rigid, but it is linearized with weights $\beta + i/d$; β coming from the weight of the trivialization of the pullback of $\mathcal{O}_{\mathbb{P}^1}(-1)$ in the chart over 0, i/d from the section $1/z^i$. Notice that if one were to reproduce this computation using a local coordinate over ∞ instead, the corresponding weights would now be $(\beta+1) - (d-i)/d$, which are exactly the same.

2.4 The Euler Class of the Normal Bundle to the Fixed Loci

The standard way to carry out this computation is to analyze the deformation long exact sequence, and identify the fiber of the normal bundle to a fixed locus at a particular moduli point to the moving part (the part where the $\mathbb{C}^*$ action does not lift trivially) of the tangent space to the moduli space (corresponding to the space of first order deformations of the admissible cover in question). It is shown in [ACV01], page 3561, that the deformation theory of admissible covers corresponds exactly to the deformation theory of the base, genus 0, twisted curve. The reason for this is that admissible covers are étale covers (in fact principal S_d-bundles) of the base twisted curve.

Deformations of a genus 0 nodal twisted curve are described as follows: first of all, we can deal with one node at a time. For one given node, there are two different potential contributions:

- the contribution from moving the node on the main $\mathbb{P}^1$. Doing this infinitesimally means moving along the tangent space to the attaching point on the main $\mathbb{P}^1$. Again, the bundle with fiber the tangent space over a given point of $\mathbb{P}^1$ is a trivial bundle, but in equivariant cohomology it can have a purely equivariant first Chern class, according to the linearization of the fibers. In our particular case, the tangent bundle has weight 1 over 0 and -1 over ∞, thus producing a contribution of $\hbar$ for moving a node around 0, of $-\hbar$ for moving a node around ∞;

- the contribution from smoothing the node. It corresponds to the first Chern class of the tensor product of the tangent spaces at the attaching points of the two curves. Again, we get a $\pm\hbar$ contribution from the point on the main $\mathbb{P}^1$; the other attaching point x, on the other hand, contributes, by definition, a $-\psi_x$ class.

An excellent reference for these kind of computations is [HKK$^+$03], chapters 23 to 29.

CHAPTER 3

TOPOLOGICAL QUANTUM FIELD THEORIES

A Topological Quantum Field Theory (TQFT) is a geometric structure whose origin lies in physics. It is a toy model for ordinary quantum field theories.

Roughly, the physical idea of a quantum field theory is the following: at any moment t in time, a physical system is described by an n-dimensional manifold X_t embedded in the appropriate slice of space-time. To any physical system, i.e., manifold, is associated a vector space H_{X_t} of all possible states of the system. If we now let time flow from t to s, the manifold deforms continuously to a new manifold X_s. The continuous deformation creates an $(n+1)$-dimensional manifold W that makes X_t and X_s cobordant. During the flow of time, and the W-evolution of the system, also any possible initial state of the system evolves to a new state. We thus associate to W a morphism between the state spaces

$$\phi_W : H_{X_t} \to H_{X_s},$$

called the *evolution operator*.

In order for this structure to make physical sense, we have to impose a condition of temporal consistency: it should not make any difference if we allow the system to flow from time t to time s, or if we first let it flow to some intermediate time r, observe it, and then let it flow form r to s. This condition translates to the fact that the evolution operator from time t to time s must be the composition of the two intermediate operators. A schematic illustration is depicted in Figure 3.1.

The simplification brought by the word "topological" means that the physical quantities, i.e. , the state spaces and the evolution operators, must depend only on the topology of the corresponding geometric objects.

From a strictly mathematical point of view, we can describe a TQFT as a functor between two appropriate categories. In particular, we will be restricting our attention to theories where the manifolds are one dimensional, the cobordisms two dimensional.

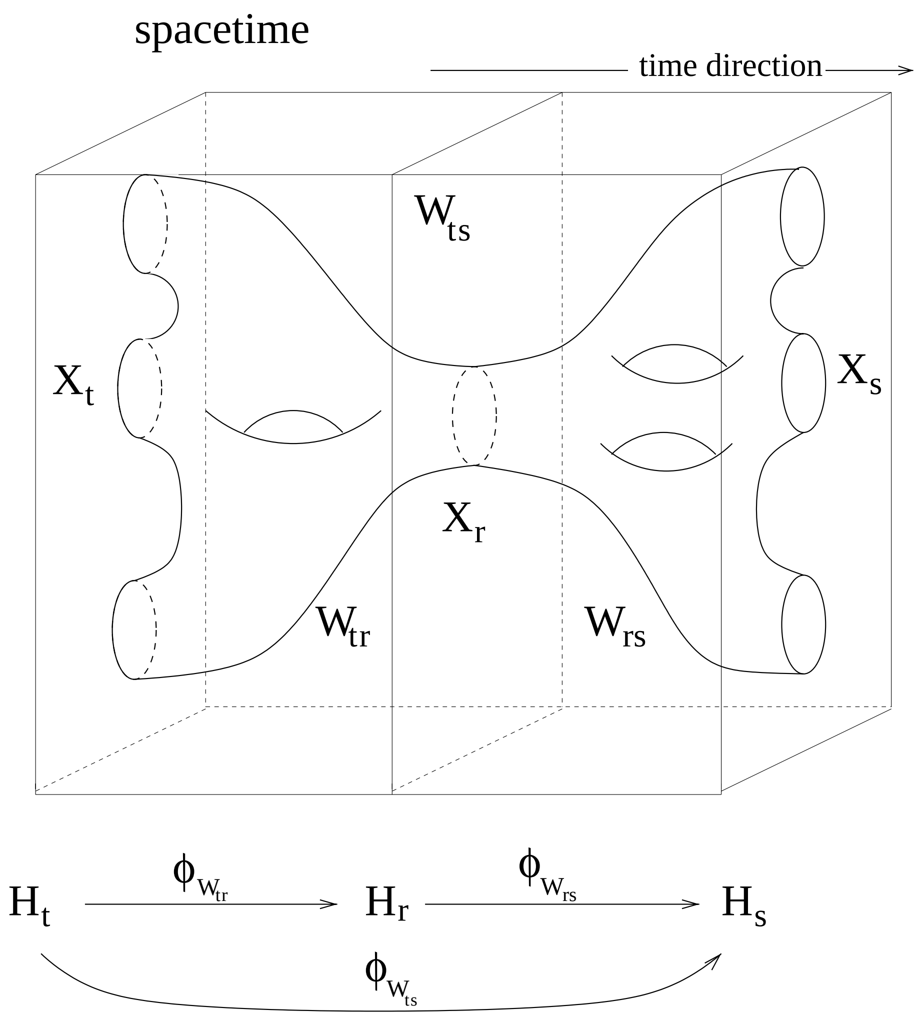


Figure 3.1. Schematic depiction of a QFT.

The simplicity of the topology in this case is reflected in the simplicity of such TQFT's. Further, a strict connection with Frobenius algebras is observed.

From the point of view of geometry, a TQFT is an extremely rigid and powerful structure that organizes topological invariants of surfaces that "behave well" with respect to gluing.

An elementary but extremely accurate and mathematically rigorous reference for this material is the book by Joachim Kock [Koc03].

3.1 (1+1)Topological Quantum Field Theories

Definition 10 *A (1+1) dimensional topological quantum field theory is a functor of tensor categories:*

$$\mathcal{T} : \boldsymbol{2Cob} \longrightarrow \boldsymbol{Vect}.$$

On the right hand side, we have the familiar category of vector spaces over a field k.

Remark: it is possible to generalize slightly to the category of modules over a commutative ring R; since no particular conceptual difference is introduced, but the usual terminology becomes slightly dishonest, we choose to give the definitions in the "classical" setting. Let us now describe the category **2Cob**:

objects: objects are one-dimensional oriented closed manifolds, i.e., finite disjoint unions of oriented circles.

morphisms: morphisms are (equivalence classes of) oriented cobordisms between two objects. We can think of them as oriented topological surfaces with oriented boundary components.

composition: we compose two morphisms by simply concatenating them; equivalently, we glue negatively oriented boundary components of one surface to positively oriented boundary components of the other.

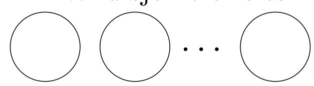
tensor structure: the tensor operation is disjoint union.

3.1.1 Objects

Since all objects in **2Cob** are generated from S^1 by "tensoring", the vector space $H := \mathcal{T}(S^1)$ plays a special role, and it is called the *Hilbert space* of the TQFT.

The functor is now completely described on all objects simply because the tensor operation needs to be respected:

n disjoint circles



$\xrightarrow{\mathcal{T}} \quad H^{\otimes n}$

3.1.2 Morphisms

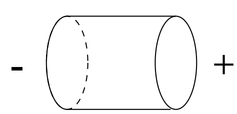
Let W be an oriented cobordism with m negatively oriented boundary components (input holes) and n positively oriented boundary components (output holes). The TQFT associates to it a linear map

$$\mathcal{T}(W) : H^{\otimes m} \longrightarrow H^{\otimes n}.$$

Composition of functions corresponds to concatenation of cobordisms, as illustrated in Figure 3.1.

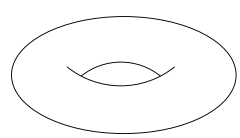
First observations:

identity:



$\xrightarrow{\mathcal{T}}$ Id: $H \to H$

torus:



$\xrightarrow{\mathcal{T}} \quad \dim_k H$

All topological surfaces can be decomposed into discs, annuli, and pairs of pants. Therefore, the structure of a TQFT is completely determined if it is described on these basic building blocks. There are many minimal choices for a set of generators for all morphisms. One that is particularly tuned to our applications is illustrated in Figure 3.2.

3.1.2.1 Tensor Notation

It is convenient, for explicit computations, to familiarize ourselves with tensor notation for TQFT's. Let us explicitly choose a basis $\mathrm{e}_1, \ldots, \mathrm{e}_r$ for the Hilbert space H, and let us

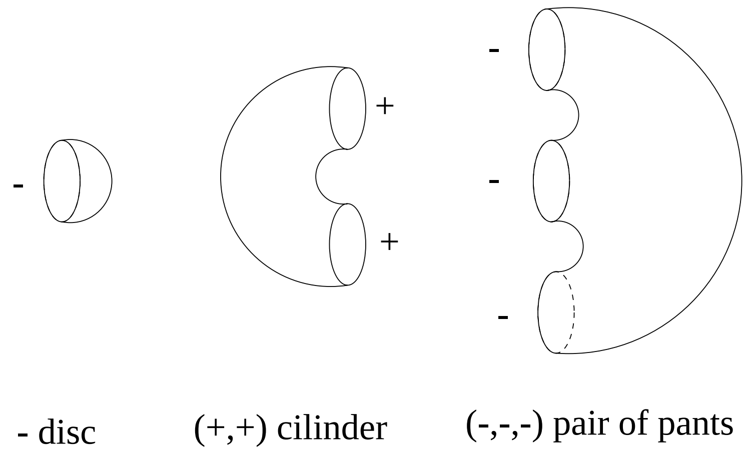


Figure 3.2. Generators for morphisms in **2Cob**.

denote the dual basis by $e^1, \ldots, e^r$. Let W be a genus g cobordism from m to n circles. Then the map

$$\mathcal{T}(W) : H^{\otimes m} \to H^{\otimes n}$$

can be thought of as a vector in $(H^*)^{\otimes m} \otimes H^{\otimes n}$. We will denote by

$$\Gamma(W)^{j_1,\ldots,j_n}_{i_1,\ldots,i_m}$$

the coefficient of $\mathcal{T}(W)$ in the direction of the basis element $e^{i_1} \otimes \cdots \otimes e^{i_m} \otimes e_{j_1} \otimes \cdots \otimes e_{j_n}$. That is,

$$\mathcal{T}(W) = \sum \Gamma(W)^{j_1,\ldots,j_n}_{i_1,\ldots,i_m} e^{i_1} \otimes \cdots \otimes e^{i_m} \otimes e_{j_1} \otimes \cdots \otimes \; e_{j_n},$$

as depicted in Figure 3.3.

Remark: when using tensor notation, it is possible to glue boundary circles one at a time, or even glue a negative and a positive boundary circles on the same surface. It is then immediate to see that if you have a genus $g-1$ cobordism W from 1 circle to 1 circle, and you glue the boundary circles together, you obtain that the genus g empty cobordism corresponds to the trace of the linear map $\mathcal{T}(W)$.

3.1.3 Frobenius Algebras

A TQFT gives the Hilbert space H the structure of a commutative Frobenius algebra. This means it defines an associative and commutative multiplication " $\cdot$" and an inner product (also called the metric of the TQFT) "$<,>$" on H such that

$$< h_1 \cdot h_2, h_3 >=< h_1, h_2 \cdot h_3 > \tag{3.1}$$

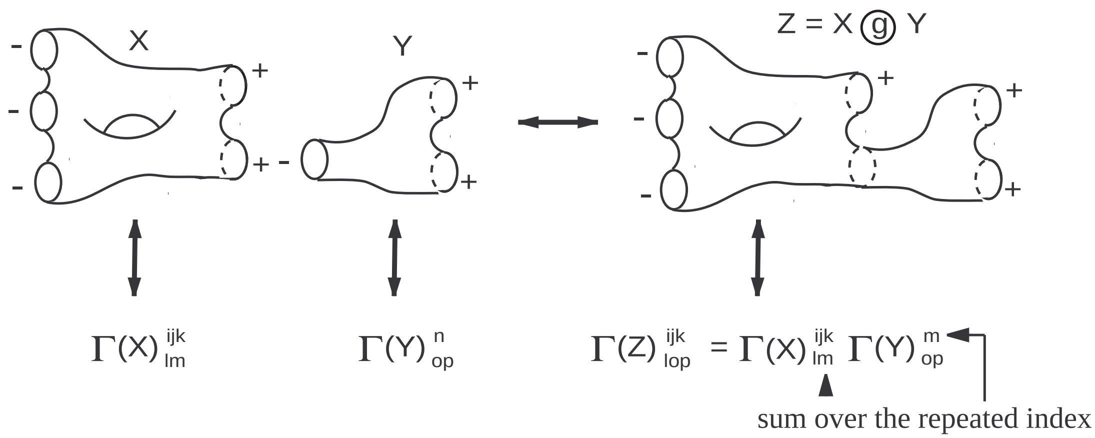


Figure 3.3. Gluing in tensor notation.

holds for all h_1,h_2,h_3 in the Hilbert space H. It is easy to see how the structure is induced: multiplication is the map associated to the $(-,-,+)$-pair of pants, the inner product is the scalar map associated to the $(-,-)$-annulus, as illustrated in Table 3.1. As a consequence, we see immediately that the cap with positively oriented boundary corresponds to the unit vector for the multiplication map just defined. Table 3.1 illustrates these and some other algebraic objects defined by a TQFT.

Notice in particular that another extremely natural set of generators for all morphisms in **2Cob** is given by the counit, the coproduct and the multiplication.

3.2 Dimension 1 Hilbert Space

This is a simple, but extremely important example, as it basically allows us to describe the structure of any semisimple Frobenius algebra (see section 3.3).

Let the Hilbert space coincide with the ground field k. Then the TQFT structure is completely determined by the assignment of a nonzero element λ in k. Notice first of all that the multiplication and the unit are forced to coincide with the multiplication and unit in k. Then, all morphisms are generated once we assign, for example, the inner product. This corresponds to choosing an element $1/\lambda \in Hom(k,k)^{\otimes 2} = k$. The element is chosen to be invertible so that the inner product is nondegenerate. Table 3.2 gives the values of the TQFT for a few of the common building blocks of the structure. Of particular importance is the genus adding operator, which allows us to increase the genus of our cobordisms.

Table 3.1. Algebraic objects defined by a TQFT.

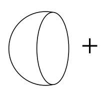
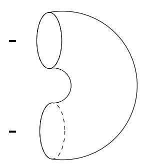
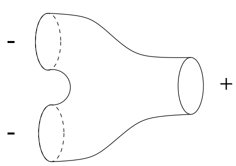
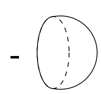
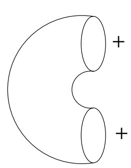
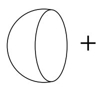
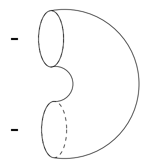
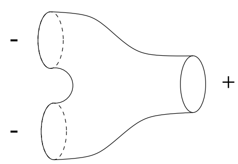
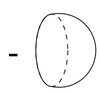
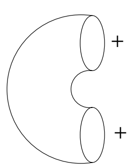
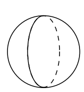
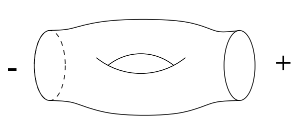
	$\xrightarrow{\mathcal{T}}$	$u : k \to H$	unit
	$\xrightarrow{\mathcal{T}}$	$<,>: H \otimes H \to k$	inner product
	$\xrightarrow{\mathcal{T}}$	$\cdot : H \otimes H \to H$	multiplication
	$\xrightarrow{\mathcal{T}}$	$u^* : H \to k$	counit
	$\xrightarrow{\mathcal{T}}$	$\Delta : k \to H \otimes H$	coproduct

Table 3.2. The structure of a TQFT with one-dimensional Hilbert space.

	$\xrightarrow{\mathcal{T}}$	1	unit
	$\xrightarrow{\mathcal{T}}$	$1/\lambda$	inner product
	$\xrightarrow{\mathcal{T}}$	1	multiplication
	$\xrightarrow{\mathcal{T}}$	$1/\lambda$	counit
	$\xrightarrow{\mathcal{T}}$	λ	coproduct
	$\xrightarrow{\mathcal{T}}$	$1/\lambda$	$< u, u >$
	$\xrightarrow{\mathcal{T}}$	λ	genus adding operator

Using this definitions, and just by decomposing appropriately any cobordism into elementary pieces, it is completely straightforward to prove the following structure result.

Lemma 3 *Let $\mathcal{T}$ be a TQFT with a one-dimensional Hilbert space. Let λ be a non-zero element in k assigned to the inner product as above. Denote by $W_m^n(g)$ a genus g surface with m input and n output holes. Identify canonically all tensor products of various copies of k and k-dual with k itself.*
Then:

$$\mathcal{T}(W_m^n(g)) = \lambda^{g+n-1}.$$

In particular:

$$\mathcal{T}(W_0^0(g)) = \lambda^{g-1}. \tag{3.2}$$

3.3 Semisimple TQFT's

Definition 11 *A TQFT $\mathcal{T}$ is semisimple if the Frobenius algebra induced on the Hilbert space H is semisimple. That is, if there is an orthonormal basis $e_1, \dots e_r$ for H such that*

$$e_i \cdot e_j = \delta_{ij} e_i.$$

An equivalent point of view is to say that $\mathcal{T}$ is a direct sum

$$\mathcal{T} = \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_r,$$

where all $\mathcal{T}_i$'s are TQFT with one-dimensional Hilbert space.

Denote by $\mathrm{e}_1, \dots, \mathrm{e}_r$ a semisimple basis for H. We can also think of e_i being the identity vector for the space H_i. Let $\mathrm{e}^1, \dots, \mathrm{e}^r$ be the dual basis. Then simisimplicity is equivalent to asking all nondiagonal coefficients to vanish:

$$\Gamma_{i_1,\dots,i_n}^{j_1,\dots,j_m}(W) = 0,$$

unless $i_1 = i_2 = \cdots = i_n = j_1 = \cdots = j_m$.

There are now r universal constants $\lambda_1, \dots, \lambda_r$ that govern the structure of the TQFT. They can be defined in many equivalent ways. Here are two equivalent descriptions that we will be using later on:

1. $1/\lambda_i$ is the image of the basis vector e_i via the counit operator.

2. λ_i is the i-th eigenvalue of the genus adding operator.

Now the following structure theorem holds:

Theorem 1 *Let $\mathcal{T}$ be a semisimple TQFT, and all notation as above. Denote by $W_m^n(g)$ a genus g surface with m input and n output holes. Then:*

$$\mathcal{T}(W_m^n(g)) = \sum_{i=1}^{r} \lambda_i^{g+n-1} \underbrace{e^i \otimes \cdots \otimes e^i}_{m \quad times} \underbrace{\otimes e_i \otimes \cdots \otimes e_i}_{n \quad times}.$$

In particular:

$$\mathcal{T}(W_0^0(g)) = \sum_{i=1}^{r} \lambda_i^{g-1}. \tag{3.3}$$

3.4 The TQFT of Hurwitz Numbers

In the early 1990s ([DW90]), the mathematical physicist Robbert Dijkgraaf noticed that a TQFT approach yields a beautiful and elegant solution to a classical mathematical problem: counting ramified and unramified covers of a topological surface.

Let $(X, x_1, \ldots, x_n)$ be an n-marked smooth topological surface. Let $\underline{\eta} = (\eta_1, \ldots, \eta_n)$ be a vector of partitions of the integer d. We define the *Hurwitz number*:

$$H_d^X(\underline{\eta}) \quad := \quad \text{weighted number of} \left\{ \begin{array}{c} degree\ d\ covers \\ C \xrightarrow{\pi} X\ such\ that: \\ \bullet\ \pi\ is\ unramified\ over\ X \setminus \{x_1, \ldots x_n\}; \\ \bullet\ \pi\ ramifies\ with\ profile\ \eta_i\ over\ x_i. \end{array} \right\}$$

The above number is weighted by the number of automorphisms of such covers.

We obtain an equivalent definition of the Hurwitz number by counting the number of homomorphisms from the fundamental group of the punctured surface $X \setminus \{x_1, \ldots x_n\}$ to the symmetric group S_d, such that the image of loops around each puncture x_i lies in the conjugacy class identified by the partition η_i; then we divide by $d!$ because we want to quotient out by the natural action by conjugation of S_d on such homomorphisms :

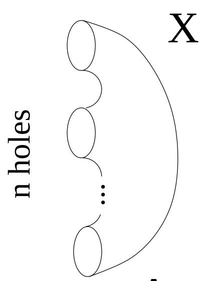
$$\boxed{H_d^X(\underline{\eta}) = \mid Hom^{(\underline{\eta})}(\pi_1(X \setminus \{x_1, \ldots x_n\}), S_d)/S_d. \mid}$$

We define the TQFT $\mathcal{D}$ as follows:

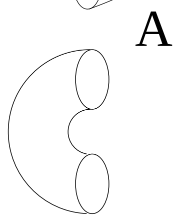
1. the ground field is $\mathbb{C}$;

2. the Hilbert space is $H = \bigoplus_{\eta \vdash d} \mathbb{C}e_\eta$;

3. morphisms are assigned according to the prescription:



$$\stackrel{\mathcal{D}}{\mapsto} \quad \mathcal{D}(X) \; : \quad \begin{array}{ccc} H^{\otimes n} & \longrightarrow & \mathbb{C} \\ \mathrm{e}_{\eta_1} \otimes \ldots \otimes \mathrm{e}_{\eta_n} & \mapsto & H_d^X(\underline{\eta}). \end{array}$$



$$\stackrel{\mathcal{D}}{\mapsto} \quad \mathcal{D}(A) = \sum \mathfrak{z}(\eta) \mathrm{e}_\eta \otimes \mathrm{e}_\eta.$$

The coefficient $\mathfrak{z}(\eta)$ is just a combinatorial factor that makes things "work". If $\eta = ((\eta^1)^{m_1} \ldots (\eta^k)^{m_k})$, then

$$\mathfrak{z}(\eta) := \prod (\eta^i)^{m_i} m_i!$$

Theorem 2 (Dijkgraaf) *The above assignment defines a semisimple TQFT $\mathcal{D}$. Let η be a partition of d, representing a conjugacy class of the symmetric group, and let h be an element in this conjugacy class. Via the identification:*

$$e_\eta = \frac{1}{d!} \sum_{g \in S_d} g^{-1} h g,$$

the Hilbert space is isomorphic, as a Frobenius algebra, to the class algebra of the symmetric group in d letters, $\mathcal{Z}(\mathbb{C}[S_d])$.

A semisimple basis is indexed by irreducible representations ρ of S_d. Let ρ be such a representation and $\mathcal{X}_\rho$ its character function, then:

$$e_\rho = (dim\rho) \sum_{\eta \vdash d} \mathcal{X}_\rho(\eta) e_\eta.$$

This allows Dijkgraaf to recover the classical Burnside formula, expressing the number of unramified covers of a genus g curve:

$$\boxed{\sum_\rho \left(\frac{d!}{\dim \rho} \right)^{2g-2}.} \tag{3.4}$$

3.5 Weighted TQFT's

A weighted TQFT contains some extra structure with respect to an ordinary TQFT. In simple terms, every cobordism comes equipped with a sequence of weights, or levels. When you concatenate two cobordisms, you add the levels componentwise. We are in particular interested in the theory with 2 levels.

Define the category $\mathbf{2Cob}^{k_1,k_2}$ as follows:

1. Objects and tensor structure are the same as in **2Cob**.

2. Morphisms are given by triples (W, k_1, k_2), where W is an oriented cobordism as in **2Cob**, k_1, k_2 are two integers called levels.

3. Composition of morphisms consists in concatenating the cobordisms and adding the levels componentwise.

We also generalize the target category so as to give the definition that we need for Chapter 4. Let R be a commutative ring with 1, and denote by **FRMod** the category of free R modules. The dimension of a vector space is substituted by the rank of a module, and everything carries through without any difficulty.

Definition 12 *A weighted TQFT is a functor of tensor categories:*

$$\mathcal{WT} : \mathbf{2Cob^{k_1,k_2}} \longrightarrow \mathbf{FRMod}.$$

It is immediate that if we restrict our attention to only cobordisms with weight $(0, 0)$, we obtain an ordinary TQFT. More generally, there are a $\mathbb{Z} \times \mathbb{Z}$ worth of ordinary TQFT embedded in a weighted TQFT. Denote by $\mathcal{X}$ the Euler characteristic of a cobordism W. For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, restricting the weighted TQFT to cobordisms with level

$$(a\mathcal{X}, b\mathcal{X})$$

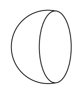
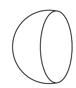
yields an ordinary TQFT.

3.5.1 Generation Results

There are several possible ways to generate a weighted TQFT. A particularly natural one consists in generating the level $(0, 0)$ TQFT, and then giving natural operators that allow one to shift the levels. These elements can be chosen to be, for example, the cylinders with weight $(\pm 1, 0)$ and $(0, \pm 1)$. These operators change the levels of the cobordisms without altering its topology. An equivalent, and equally natural choice, is given by the caps, as illustrated in Figure 3.4.

In particular, it is immediate to see that A (resp.C) is the inverse of B (resp.D) in the level $(0,0)$ Frobenius algebra. Hence the following generation result.

Theorem 3 (Bryan-Pandharipande, [BP04] 4.1) *A weighted TQFT $\mathcal{WT}$ is uniquely determined by a commutative Frobenius algebra over k for the level $(0,0)$ theory and by two distinguished invertible elements in the Frobenius algebra:*

$\mathcal{WT}\Big($  $+$, $(-1,0)\Big)$, $\mathcal{WT}\Big($  $+$, $(0,-1)\Big)$.

3.5.2 Semisimple Weighted TQFT

A weighted TQFT of rank r is semisimple if all the non-zero tensors in the theory are diagonal. This is equivalent to asking that all embedded ordinary TQFT's are semisimple (possibly with different semisimple bases). Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of the level $(0,0)$ genus adding operator. Let $\mu_1, \ldots, \mu_r$ be the eigenvalues of the level $(-1,0)$ annulus, and $\overline{\mu}_1, \ldots, \overline{\mu}_r$ be the eigenvalues for the level $(0,-1)$ annulus, as illustrated in Figure 3.5.

Theorem 4 (Bryan-Pandharpiande,[BP04],5.2) *Let $\mathcal{WT}$ be a semisimple TQFT. Denote by $W_m^n(g|k_1,k_2)$ a cobordism of genus g between m input and n output holes, of level (k_1,k_2). Then:*

$$\mathcal{T}(W_m^n(g|k_1,k_2)) = \sum_{i=1}^{r} \lambda_i^{g+n-1}\mu_i^{-k_1}\overline{\mu}_i^{-k_2} \underbrace{e^i \otimes \cdots \otimes e^i}_{m \quad times} \underbrace{\otimes e_i \otimes \cdots \otimes e_i}_{n \quad times}.$$

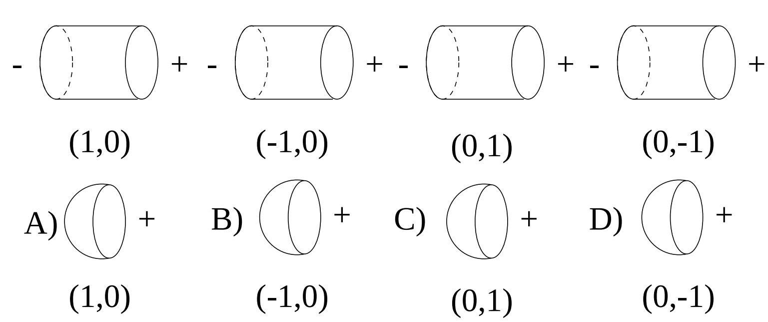


Figure 3.4. Level changing objects.

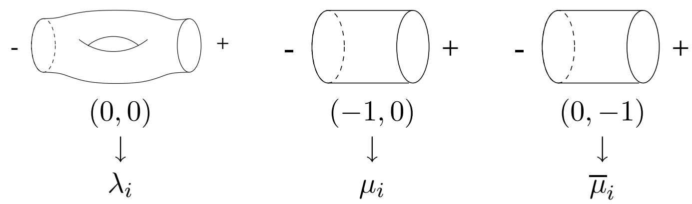


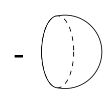
Figure 3.5. The genus-adding and the level-changing operators.

In particular:

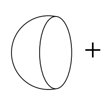
$$\mathcal{T}(W_0^0(g|k_1, k_2)) = \sum_{i=1}^{r} \lambda_i^{g-1} \mu_i^{-k_1} \overline{\mu}_i^{-k_2}. \tag{3.5}$$

Observation: the following equivalent definitions can be given for the quantities λ_i, μ_i and $\overline{\mu}_i$. Denote by $\mathrm{e}_1, \ldots, \mathrm{e}_r$ the vectors of a semisimple basis for the weighted TQFT $\mathcal{WT}$:

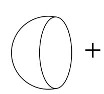
- λ_i^{-1} is the value of the level $(0, 0)$ counit on e_i:

$\mathcal{WT}$ ( $(0,0)$) $(\mathrm{e}_i) = \lambda_i^{-1}.$

- μ_i is the coefficient of e_i in the level $(-1, 0)$ +disc vector:

$\mathcal{WT}$ ( $(-1,0)$) $= \sum \mu_i \mathrm{e}_i.$

- $\overline{\mu}_i$ is the coefficient of e_i in the level $(0, -1)$ +disc vector:

$\mathcal{WT}$ ( $(0,-1)$) $= \sum \overline{\mu}_i \mathrm{e}_i.$

CHAPTER 4

THE WEIGHTED TQFT OF ADMISSIBLE COVERS

In [BP03], Jim Bryan and Rahul Pandharipande have given a TQFT structure to the local Gromov-Witten theory of curves embedded in a Calabi-Yau threefold. This TQFT (that we call $\mathcal{BP}$) is a one parameter deformation of the classical Dijkgraaf TQFT $\mathcal{D}$: the ground field $\mathbb{C}$ is replaced by the formal power series ring $\mathbb{C}[[t]]$; the coefficients of $\mathcal{BP}$ are generating functions for local Gromov-Witten invariants, that evaluated at $t = 0$ reduce to the corresponding Hurwitz numbers in $\mathcal{D}$. In their recent paper [BP04], the two authors have embedded this theory in an even richer structure, and defined a weighted TQFT encoding the equivariant local Gromov-Witten invariants of any curve that is the zero section of a rank two vector bundle.

In this chapter, we develop a parallel theory to [BP04]. We define analogous intersection numbers on moduli spaces of admissible covers and show that they satisfy similar gluing properties. We construct a weighted TQFT and explicitly compute it in low degrees. The two theories are very much related to one another, possibly in a very geometric way. The moduli spaces of admissible covers however, seem to be more natural than the moduli spaces of stable maps for this kind of intersection problem.

4.1 The Admissible Covers Closed Invariants

Let X be a smooth, irreducible, projective curve of genus g, and $N = L_1 \oplus L_2$ a rank 2 vector bundle on X. The torus $T = \mathbb{C}^* \times \mathbb{C}^*$ acts naturally on N: the first coordinate scales (with weight one) the fibre of L_1, the second coordinate scales the fibre of L_2.

We are interested in the following class of intersection numbers:

$$A^h_d(N) := \int_{\overline{Adm}_{h \xrightarrow{d} X}} e(-R^\bullet \pi_* f^*(L_1 \oplus L_2)),$$

where:

- $\overline{Adm}_{h \xrightarrow{d} X}$ denotes the space of (possibly disconnected) admissible covers of degree d of the curve X, with all ramification simple;

- integration means equivariant push-forward to the class of a point.

- e is the equivariant Euler class of the virtual bundle in question.

- π is the universal family over the space of admissible covers.

- f is the universal cover map followed by the canonical contraction map to X, as in section 1.2.

By [BP01], this integral only depends on the genus g of the curve X and on the degrees k_1 and k_2 of the line bundles L_1 and L_2. In our forthcoming TQFT formulation it will be useful to emphasize this fact, so we choose to denote the above invariants:

$$A^h_d(N) = A^h_d(g|k_1, k_2).$$

We consider these invariants for all genera h, and organize them in generating function form as follows:

$$A_d(g|k_1, k_2) := \sum_{h \in \mathbb{Z}} u^{\star(h)} A^h_d(g|k_1, k_2). \tag{4.1}$$

The appropriate exponent for the generating function is defined:

$$\star(h) = dim(\overline{Adm}_{h \xrightarrow{d} X}) = 2h - 2 + d(2 - 2g).$$

Let us express the above equivariant integrals in terms of ordinary (meaning non equivariant) integrals of ordinary Chern classes and of the equivariant parameters s_1, s_2.

First off, the (equivariant) Euler class of the bundle $-R^\bullet \pi_* f^*(L_1 \oplus L_2)$ splits as the product of the Euler classes of the two summands.

By Riemann-Roch,the virtual bundle $-R^\bullet \pi_* f^*(L_i)$ has rank $r_i = h - 1 - dk_i$. Hence,

$$e^{eq}(-R^\bullet \pi_* f^*(L_1 \oplus L_2)) = c^{eq}_{r_1}(-R^\bullet \pi_* f^*(L_1)) c^{eq}_{r_2}(-R^\bullet \pi_* f^*(L_2)).$$

Next, the first factor of the two dimensional torus T acts on L_1 with weight 1:

$$c^{eq}_{r_1}(-R^\bullet \pi_* f^*(L_1)) = \sum_{b_1=0}^{\infty} s_1^{r_1 - b_1} c_{b_1}(-R^\bullet \pi_* f^*(L_1)).$$

Similarly,

$$c^{eq}_{r_2}(-R^{\bullet}\pi_* f^*(L_2)) = \sum_{b_2=0}^{\infty} s_2^{r_2-b_2} c_{b_2}(-R^{\bullet}\pi_* f^*(L_2)).$$

Hence,

$$e^{eq}(-R^{\bullet}\pi_* f^*(L_1 \oplus L_2)) = \sum_{b_1,b_2} s_1^{r_1-b_1} s_2^{r_2-b_2} c_{b_1}(-R^{\bullet}\pi_* f^*(L_1)) c_{b_2}(-R^{\bullet}\pi_* f^*(L_2)).$$

When pushing forward to the class of a point, the nonvanishing contributions come from summands such that

$$b_1 + b_2 = dim(\overline{Adm}_{h \xrightarrow{d} X}) = 2h - 2 + d(2 - 2g).$$

From this we can write:

$$A_d^h(g|k_1,k_2) = \sum_{b_1+b_2=dim(\overline{Adm}_{h \xrightarrow{d} X})} s_1^{r_1-b_1} s_2^{r_2-b_2} A_d^{b_1,b_2}(g|k_1,k_2),$$

where we have used the notation:

$$A_d^{b_1,b_2}(g|k_1,k_2) := \int_{\overline{Adm}_{h \xrightarrow{d} X}} c_{b_1}(-R^{\bullet}\pi_* f^*(L_1)) c_{b_2}(-R^{\bullet}\pi_* f^*(L_2)).$$

This notation is legitimate because given a pair of positive numbers b_1 and b_2 there is at most one value of h for which the above expression can be nonzero. Finally, putting these results in generating function form and after some algebraic manipulation, we obtain:

$$A_d(g|k_1,k_2) := \sum_{b_1+b_2=0}^{\infty} u^{b_1+b_2} s_1^{r_1-b_1} s_2^{r_2-b_2} A_d^{b_1,b_2}(g|k_1,k_2). \tag{4.2}$$

This shows that the partition function for our invariants is a Taylor series in u, whose coefficients are rational functions in s_1 and s_2. It is easy to see that the degree of these rational functions is independent from h. It is equal to $r_1 + r_2 - b_1 - b_2 = d(2g-2)$.

4.2 The Relative Invariants

Let $(X, x_1, \ldots, x_r)$ be a smooth projective curve of genus g with r distinct marked points. Let $\underline{\eta} = (\eta_1, \ldots, \eta_r)$ denote a vector of partitions of the integer d. We interpret η_i as prescribing a ramification condition over the point x_i. Let us consider the moduli space

$$\overline{Adm}_{h \xrightarrow{d} X,(\eta_1 x_1,\cdots,\eta_r x_r)}$$

of genus h, degree d, admissible covers of the curve X, with ramification of type η_i over the points x_i, and at most simple ramification elsewhere.

Let $N = L_1 \oplus L_2$ be the total space of a rank 2 vector bundle on X, on which the two dimensional torus T acts naturally. For $i = 1, 2$, the degree of the line bundle L_i is denoted k_i.

The relative invariants are defined as follows:

$$A_d^h(g|k_1, k_2)_{\underline{\eta}} := \int_{\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 x_1, \cdots, \eta_r x_r)}} e(-R^\bullet \pi_* f^*(L_1 \oplus L_2)).$$

As the genus h varies, we organize them in generating function form:

$$A_d(g|k_1, k_2)_{\underline{\eta}} := \sum_{h \in \mathbb{Z}} u^{\star(h)} A_d^h(g|k_1, k_2)_{\underline{\eta}}. \tag{4.3}$$

The appropriate exponent for the generating function is similarly defined:

$$\star(h) = dim(\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 x_1, \cdots, \eta_r x_r)}) = 2h - 2 + d(2 - 2g - r) + \sum_{i=1}^{r} \ell(\eta_i).$$

Again, we can express these invariants in terms of nonequivariant integrals. Let $h \in \mathbb{Z} \cup \phi$ be a function of b_1, b_2 determined by the equation

$$b_1 + b_2 = dim(\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 x_1, \cdots, \eta_r x_r)}) = 2h - 2 + d(2 - 2g - r) + \sum_{i=1}^{r} \ell(\eta_i).$$

Define

$$A_d^{b_1, b_2}(g|k_1, k_2)_{\underline{\eta}} := \int_{\overline{Adm}_{h \xrightarrow{d} X, (\eta_1 x_1, \cdots, \eta_r x_r)}} c_{b_1}(-R^\bullet \pi_* f^*(L_1)) c_{b_2}(-R^\bullet \pi_* f^*(L_2)).$$

Then the relative invariants are Taylor series in u with coefficients rational functions in s_1, s_2, given by:

$$A_d(g|k_1, k_2)_{\underline{\eta}} := \sum_{b_1 + b_2 = 0}^{\infty} u^{b_1 + b_2} s_1^{r_1 - b_1} s_2^{r_2 - b_2} A_d^{b_1, b_2}(g|k_1, k_2)_{\underline{\eta}}. \tag{4.4}$$

4.3 The Weighted TQFT

Our goal is to construct a weighted TQFT $\mathcal{U}$, whose structure coefficients encode the invariants just presented.

The ground ring is defined to be $R = \mathbb{C}[[u]](s_1, s_2)$.

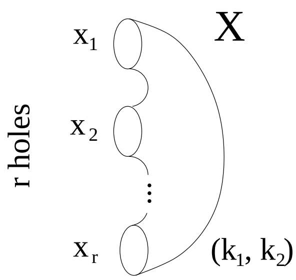
The Hilbert space of the theory is a free R-module of rank equal to the number of partitions of the integer d. A privileged basis will be indexed by such partitions η.

$$\mathcal{H} = \bigoplus_{\eta \vdash d} R e_\eta.$$

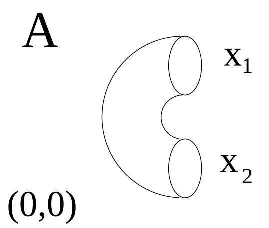
We denote the dual space by $\mathcal{H}^*$, and the dual basis vectors by e^η.

In order to construct our TQFT we need to reason topologically. We think of the marked points on a curve $(X, x_1, \ldots, x_{r+s})$ as of punctures that we can "enlarge" into loops. We can assign positive or negative orientation to such loops, and arrange the negatively oriented loops $x_1, \ldots, x_r$ to the left, the positively oriented to the right (after relabelling $x_{r+i} = y_i$). We now have an oriented cobordism.

To completely determine the structure of the theory we define the scalar maps associated to arbitrary cobordisms into the empty set, and the coproduct, that allows us to "move" boundary components from the left to the right.



$$\overset{\mathcal{U}}{\mapsto} \quad \begin{array}{cccc} \mathcal{U}(X) & : & H^{\otimes r} & \longrightarrow & \mathbb{C}[[u]](s_1, s_2) \\ & & \mathrm{e}_{\eta_1} \otimes \ldots \otimes \mathrm{e}_{\eta_r} & \mapsto & A_d(g|k_1, k_2)_{\underline{\eta}}. \end{array}$$



$$\overset{\mathcal{U}}{\mapsto} \quad \mathcal{U}(A) = \textstyle\sum_{\eta \vdash d} \mathfrak{z}(\eta)(s_1 s_2)^{\ell(\eta)} \mathrm{e}_\eta \otimes \mathrm{e}_\eta.$$

The combinatorial factor $\mathfrak{z}(\eta)$ is defined in page 15.

In practical terms, it is often very convenient to adopt the conventional riemannian geometry tensor notation. If $X = (X, x_1, \ldots, x_r, y_1, \ldots, y_s | k_1, k_2)$ represents a cobordism of genus g and level k_1, k_2 from r circles to s circles, then $\mathcal{U}(X)$ is an element of $(\mathcal{H}^*)^{\otimes r} \otimes \mathcal{H}^{\otimes s}$. We denote by

$$A_d(g|k_1, k_2)^{\mu_1, \ldots, \mu_s}_{\eta_1, \ldots, \eta_r}$$

the coordinate of $\mathcal{U}(X)$ in the direction of the basis element $\mathrm{e}^{\eta_1} \otimes \cdots \otimes \mathrm{e}^{\eta_r} \otimes \mathrm{e}_{\mu_1} \otimes \cdots \otimes \mathrm{e}_{\mu_s}$.

With this notation, the coproduct gives the following formula for raising and lowering indices:

$$A_d(g|k_1, k_2)^{\mu_1, \ldots, \mu_s}_{\eta_1, \ldots, \eta_r} = \left(\prod_{i=1}^{s} \mathfrak{z}(\mu_i)(s_1 s_2)^{\ell(\mu_i)} \right) A_d(g|k_1, k_2)_{\eta_1, \ldots, \eta_r, \mu_1, \ldots, \mu_s}. \tag{4.5}$$

4.4 Proof of TQFT Structure

Proving that the functor $\mathcal{U}$ just defined is indeed a weighted TQFT amounts to verifying the following three statements:

identity: the tensor associated to the level $(0,0)$ trivial cobordism from the circle to the circle is the identity morphism of the Hilbert space $\mathcal{H}$.

gluing two curves: for any two vectors $\underline{\eta}, \underline{\mu}$ of partitions of d, and integers satisfying $g = g' + g''$, $k_1 = k_1' + k_1''$, $k_2 = k_2' + k_2''$,

$$A_d(g|k_1,k_2)^{\mu_1,\dots,\mu_s}_{\eta_1,\dots,\eta_r} = \sum_{\nu \vdash d} A_d(g'|k_1',k_2')^{\nu}_{\eta_1,\dots,\eta_r} A_d(g''|k_1'',k_2'')^{\mu_1,\dots,\mu_s}_{\nu}. \tag{4.6}$$

self-gluing: for any vector of partitions $\underline{\eta}$, and integers g, k_1, k_2,

$$A_d(g+1|k_1,k_2)_{\eta_1,\dots,\eta_r} = \sum_{\nu \vdash d} A_d(g|k_1,k_2)^{\nu}_{\eta_1,\dots,\eta_r,\nu}. \tag{4.7}$$

4.4.1 Identity

This fact is easily proven. One very clever way to do it, which is pursued in [BP03], is to notice that the degree 0 coefficients in our TQFT agree with the classical TQFT of Hurwitz numbers constructed by Dijkgraaf in [DW90]. The vanishing of all higher degree terms is then obtained as a straightforward consequence of the gluing laws.

However, the coefficients of the trivial cobordism can also be obtained directly via a localization computation, which we will carry out explicitly in section 4.5.2 as a good warm-up for the more complex computations to come.

4.4.2 Gluing Two Curves

In order to minimize the burden of bookkeeping, we will prove the result when $r = s = 0$ (i.e. , the resulting glued curve is not marked). In the general case, the proof follows exactly the same steps, and all the extra indices are simply carried along for the ride.

Consider a one parameter family of genus g curves W, and the corresponding map to the moduli space,

$$\begin{array}{ccccc} & & W & & \\ & & \downarrow & & \\ \varphi: & & \mathbb{A}^1 & \to & \overline{M}_g, \end{array}$$

such that:

- the central fiber
$$W_0 = X_1 \bigcup_{b_1=b_2} X_2$$
is a nodal curve obtained by attaching at a point two smooth curves of genus g' and g'' (with $g' + g'' = g$);

- all other fibers $W_s, s \neq 0$, are smooth curves of genus g.

Consider the moduli space $\overline{Adm}_{h \xrightarrow{d} g}$ of admissible covers of a genus g curve by a genus h curve, all ramification simple. By [ACV01], there is a flat morphism

$$\overline{Adm}_{h \xrightarrow{d} g} \to \overline{M}_g,$$

We can construct the following cartesian diagram:

$$\begin{array}{ccccc} \mathcal{A}_s = \overline{Adm}_{h \xrightarrow{d} W_s} & \hookrightarrow & \mathcal{A} & \to & \overline{Adm}_{h \xrightarrow{d} g} \\ \downarrow & & \downarrow & & \downarrow \\ \{s\} & \hookrightarrow & \mathbb{A}^1 & \to & \overline{M}_g \end{array}$$

The stack $\mathcal{A}$ must be thought as of the stack of relative admissible covers of the family W. For $s \neq 0$, we obtain admissible covers of a smooth genus g curve; for $s = 0$, we recover admissible covers of the nodal curve W_0.

Recall that, by section 1.3.1.1,

$$[\overline{Adm}_{h \xrightarrow{d} W_0}] = \sum_{\nu \vdash d} \mathfrak{z}(\nu) \sum_{h_1,h_2} [\overline{Adm}_{h_1 \xrightarrow{d} X_1,(\nu b_1)}] \times [\overline{Adm}_{h_2 \xrightarrow{d} X_2,(\nu b_2)}],$$

where:

- $h_1 + h_2 + \ell(\nu) - 1 = h$;

- $dim(\overline{Adm}_{h_1 \xrightarrow{d} X_1,(\nu b_1)}) + dim(\overline{Adm}_{h_2 \xrightarrow{d} X_2,(\nu b_2)}) = dim(\overline{Adm}_{h \xrightarrow{d} W_0})$.

It is possible to construct two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on W, with the following properties:

1. $\mathcal{L}_i$ restricted to any fiber W_s is a line bundle $L_{i,s}$ of degree k_i.

2. Over the central fiber W_0, $\mathcal{L}_i$ restricts to a line bundle $L'_{i,s}$ of degree k'_i on X_1, and restricts to a line bundle $L''_{i,s}$ of degree k''_i on X_2.

3. $\mathbb{C}^*$ acts naturally on $\mathcal{L}_i$ by scaling the fibers (with weight one).

Consider the following diagram:

$$\begin{array}{ccccc} \mathcal{U}_{\mathcal{A}} & \stackrel{f}{\to} & \mathcal{W} & \to & W \\ \pi\downarrow & \swarrow & & & \\ \mathcal{A} & & & & \end{array}$$

$\mathcal{U}_{\mathcal{A}}$ is the universal family of the moduli space $\mathcal{A}$, $\mathcal{W}$ is the universal target and f the universal admissible cover map.

The pull-push

$$\mathcal{I} = -R^{\bullet}\pi_* f^*(\mathcal{L}_1 \oplus \mathcal{L}_2)$$

is a virtual bundle of virtual rank $r = 2g - 2 - d(k_1 + k_2)$.

By the flatness of the family $\mathcal{A}$ over $\mathbb{A}^1$, the integral of the top Chern class $c_r(\mathcal{I})$ restricted to a fiber $\mathcal{A}_s$ is independent of the fiber. For $s \neq 0$, we obtain

$$\int_{\overline{Adm}_{h \stackrel{d}{\to} W_s}} c_r(\mathcal{I} \mid_s) = A_d^h(g|k_1, k_2) \tag{4.8}$$

To conclude our proof it suffices to establish the following claim, which consists of expanding the genus h term in equation (4.6), and lowering indices as in (4.5).

Claim 1

$$\int_{\overline{Adm}_{h \stackrel{d}{\to} W_0}} c_r(\mathcal{I} \mid_0) = \sum_{\nu \vdash d} \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1, h_2} A_d^{h_1}(g'|k_1', k_2')_\nu A_d^{h_2}(g''|k_1'', k_2'')_\nu,$$

where the second sum is over pairs of indices such that $h_1 + h_2 + \ell(\nu) - 1 = h$.

Proof: Consider the pull-back of the normalization sequence associated to the restriction of $\mathcal{L}_i$ to W_0:

$$0 \to f^*(L_{i,0}) \to f^*(L'_{i,0}) \oplus f^*(L''_{i,0}) \to f^*(L_{i,0}) \mid_{X_1 \cap X_2} \to 0.$$

This sequence yields a long exact sequence of higher direct image sheaves

$$0 \to R^0\pi_* f^*(L_{i,0}) \to R^0\pi_* f^*(L'_{i,0}) \oplus R^0\pi_* f^*(L''_{i,0}) \to R^0\pi_* f^*(L_{i,0}) \mid_{X_1 \cap X_2} \to$$

$$\to R^1\pi_* f^*(L_{i,0}) \to R^1\pi_* f^*(L'_{i,0}) \oplus R^1\pi_* f^*(L''_{i,0}) \to 0.$$

Notice that $(L_{i,0}) \mid_{X_1 \cap X_2}$ is but a skyscraper sheaf $\mathbb{C}_b$, on which $\mathbb{C}^*$ acts with weight 1.

Let us restrict our attention to a connected component of $\mathcal{A}_0$ on which the covers split as two smooth covers of genus h_1 and h_2, with ramification profile ν over the shadows of the node. Here, $f^*(L_{i,0}) \mid_{X_1 \cap X_2}$ is a trivial vector bundle of rank $\ell(\nu)$, endowed with a natural $\mathbb{C}^*$ action.

From this fact and the exact sequence above we conclude:

$$c_{r_i}(-R^\bullet \pi_* f^*(L_{i,0})) = s_i^{\ell(\nu)} c_{r_i'}(-R^\bullet \pi_* f^*(L'_{i,0})) c_{r_i''}(-R^\bullet \pi_* f^*(L''_{i,0})),$$

and finally

$$c_r(\mathcal{I} \mid_0) = (s_1 s_2)^{\ell(\nu)} c_{r'}(\mathcal{I} \mid_0') c_{r''}(\mathcal{I} \mid_0''). \tag{4.9}$$

Now cinching the claim is just a matter of putting everything together:

$$\begin{aligned}
\int_{\overline{Adm}_{h \xrightarrow{d} W_0}} c_r(\mathcal{I} \mid_0) &= \sum_\nu \mathfrak{z}(\nu) \sum_{h_1,h_2} \int_{\overline{Adm}_{h_1 \xrightarrow{d} X_1,(\nu b_1)} \times \overline{Adm}_{h_2 \xrightarrow{d} X_2,(\nu b_2)}} c_r(\mathcal{I} \mid_0) \\
&= \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1,h_2} \int_{\overline{Adm}_{h_1 \xrightarrow{d} X_1,(\nu b_1)}} c_{r'}(\mathcal{I} \mid_0') \int_{\overline{Adm}_{h_2 \xrightarrow{d} X_2,(\nu b_2)}} c_{r''}(\mathcal{I} \mid_0'') \\
&= \sum_\nu \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \sum_{h_1,h_2} A_d^{h_1}(g'|k_1', k_2')_\nu A_d^{h_2}(g''|k_1'', k_2'')_\nu \quad .
\end{aligned}$$

4.4.3 Self-gluing

The structure of the proof is very similar to the previous case. Again, we simplify the notation by assuming $r = 0$ (i.e. , the gluing corresponding to taking the trace of a linear map).

Consider a one parameter family of genus g curves W, and the corresponding map into the moduli space,

$$\begin{array}{cccc} & W & & \\ & \downarrow & & \\ \varphi: & \mathbb{A}^1 & \rightarrow & \overline{M}_g, \end{array}$$

such that:

- the central fiber

  $$W_0 = X/\{b_1 = b_2\}$$

  is a nodal curve obtained by identifying two distinct points on an irreducible smooth curve X of genus $g-1$;

- all other fibers $W_s, s \neq 0$, are smooth curves of genus g.

As before, we can construct the following cartesian diagrams:

$$\begin{array}{ccccc} \mathcal{A}_s = \overline{Adm}_{h \xrightarrow{d} W_s} & \hookrightarrow & \mathcal{A} & \rightarrow & \overline{Adm}_{h \xrightarrow{d} g} \\ \downarrow & & \downarrow & & \downarrow \\ \{s\} & \hookrightarrow & \mathbb{A}^1 & \rightarrow & \overline{M}_g \end{array}$$

and two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on W, with the following properties:

1. $\mathcal{L}_i$ restricted to any fiber W_s is a line bundle $L_{i,s}$ of degree k_i.
2. Over the central fiber W_0, $\mathcal{L}_i$ pulls back to a line bundle $L'_{i,s}$ of degree k_i on the normalization X.
3. $\mathbb{C}^*$ acts naturally on $\mathcal{L}_i$ by scaling the fibers (with weight one).

By section 1.3.1.2,

$$[\overline{Adm}_{h \xrightarrow{d} W_0}] = \sum_{\nu \vdash d} \mathfrak{z}(\nu) [\overline{Adm}_{h' \xrightarrow{d} X, (\nu b_1, \nu b_2)}],$$

with $h' + \ell(\nu) = h$

We now consider the equivariant top Chern class of the pull-push

$$\mathcal{I} = -R^\bullet \pi_* f^* (\mathcal{L}_1 \oplus \mathcal{L}_2).$$

For $s \neq 0$,

$$\int_{\overline{Adm}_{h \xrightarrow{d} W_s}} c_r(\mathcal{I} \mid_s) = A_d^h(g|k_1, k_2). \tag{4.10}$$

Again, we can show that the corresponding integral over the central fiber yields exactly the genus h expansion of the right hand side of equation (4.7).

Claim 2

$$\int_{\overline{Adm}_{h \xrightarrow{d} W_0,}} c_r(\mathcal{I} \mid_0) = \sum_{\nu} \mathfrak{z}(\nu) (s_1 s_2)^{\ell(\nu)} A_d^{h'}(g-1|k_1, k_2)_{\nu,\nu} \quad ,$$

where $h' + \ell(\nu) = h$.

Proof: As in the previous paragraph, after chasing the normalization sequence for the curve W_0, we obtain, over a connected component of $\mathcal{A}_0$ characterized by covers with ramification profile ν over the shadows of the node, the following decomposition:

$$c_r(\mathcal{I}\,|_0) = (s_1 s_2)^{\ell(\nu)} c_{r'}(\mathcal{I}\,|'_0). \tag{4.11}$$

With this in hand, it is easy to conclude:

$$\begin{aligned}
\int_{\overline{Adm}_{h\xrightarrow{d}W_0}} c_r(\mathcal{I}\,|_0) &= \sum_{\nu} \mathfrak{z}(\nu) \int_{\overline{Adm}_{h'\xrightarrow{d}X,(\nu b_1,\nu b_2)}} c_r(\mathcal{I}\,|_0) \\
&= \sum_{\nu} \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} \int_{\overline{Adm}_{h'\xrightarrow{d}X,(\nu b_1,\nu b_2)}} c_{r'}(\mathcal{I}\,|'_0) \\
&= \sum_{\nu} \mathfrak{z}(\nu)(s_1 s_2)^{\ell(\nu)} A_d^{h'}(g-1|k'_1,k'_2)_{\nu,\nu} \quad .
\end{aligned}$$

4.5 Computing the Theory

In order to determine the whole weighted TQFT it is sufficient to compute a small number of invariants, as seen in theorem 3. There are many possible choices for a set of generators; we make the following choice, illustrated in Figures 3.2 and 3.4:

generators for the level $(0,0)$ TQFT:

1. the coefficients $A_d(0|0,0)_\eta$ of the open (-)disc.
2. the coefficients $A_d(0|0,0)^{\eta,\mu}$ of the (+,+) annulus.
3. the coefficients $A_d(0|0,0)_{\eta,\mu,\nu}$ associated to the (-,-,-) pair of pants.

generators for level shifting :

4. the coefficients of the Calabi-Yau caps $A_d(0|-1,0)_\eta$ and $A_d(0|0,-1)_\eta$.

Theorem 5 *The level $(0,0)$ TQFT coincides with the theory of Bryan and Pandharipande in [BP04].*

Proof: It is simple to compute independently the coefficients for the cap, which are degenerate, in the sense that only the constant term of the series is nonzero. This computation is carried out in section 4.5.1. The coefficients for the $(+,+)$-cylinder agree by definition. In section 4.5.3 we show that the coefficients for the pair of pants are the same.

We also compute the coefficients for the $(-,-)$ annulus, even if they are not on our list of generators. The explicit computation of these invariants is in fact the basis for our particular choice of co-product.

The significant difference in the theories lies in the Calabi-Yau caps. These are computed in degrees 2 and 3 by localization on the moduli spaces of admissible covers. A conjecture for the general degree is presented.

4.5.1 The Level $(0,0)$ Cap

Lemma 4 *The invariants for the level $(0,0)$ cap are:*

$$A_d(0|0,0)_\eta = \begin{cases} \frac{1}{d!(s_1s_2)^d} & \text{if } \eta = (1^d) \\ 0 & \text{if } \eta \neq (1^d). \end{cases}$$

Proof: The general strategy for computing these invariants is to first compute the connected invariants, then obtain the disconnected ones by exponentiation. We will denote the connected invariants by "bulletting" them. In this particular case, our lemma is equivalent to the statement that the only nonzero connected one-pointed invariant occurs in degree 1 and it is exactly 1.

Recall that we are free to choose two arbitrary line bundles of degree 0 to compute our invariants. We make the natural choice of two copies of the trivial bundle, and compute the following integrals:

$$\int_{\overline{Adm}^{\bullet}_{h\xrightarrow{d}\mathbb{P}^1,(\eta\infty)}} c^{eq}_{2h-2}(-R^{\bullet}\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1})).$$

If we denote by $\mathbb{E}$ the genus h Hodge bundle on our space of admissible covers, and by c the ordinary total Chern class operator, then

$$c(-R^{\bullet}\pi_* f^*(\mathcal{O}_{\mathbb{P}^1})) = \frac{c(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}))}{c(R^0\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}))} = c(\mathbb{E}^*).$$

By formula (4.4), we are then reduced to computing integrals of the form:

$$\int_{\overline{Adm}^{\bullet}_{h\xrightarrow{d}\mathbb{P}^1,(\eta\infty)}} c_{b_1}(\mathbb{E}^*)c_{b_2}(\mathbb{E}^*),$$

where $b_1 + b_2 = dim(\overline{Adm}^{\bullet}_{h\xrightarrow{d}\mathbb{P}^1,(\eta\infty)}) = 2h + d + \ell(\eta) - 2.$

Since both b_1 and b_2 must be less than or equal to h, there is really only one such integral, given by:

- $b_1 = b_2 = h$
- $d = \ell(\eta) = 1$

Further, the second condition forces $h = 0$. We have finally only one survivor:

$$A_1^\bullet(0|0,0)_{(1)} = \frac{1}{s_1 s_2}\int_{\overline{Adm}^\bullet_{0\xrightarrow{1}\mathbb{P}^1,(1\infty)}} 1 \quad = \frac{1}{s_1 s_2},$$

as desired.

4.5.2 The Annulus

Lemma 5 *The invariants for the level* $(0,0)$ *annulus are:*

$$A_d(0|0,0)_{\eta,\mu} \quad = \quad \begin{cases} \frac{1}{\mathfrak{z}(\eta)(s_1 s_2)^{\ell(\eta)}} & \text{if } \eta = \mu \\ \\ 0 & \text{if } \eta \neq \mu. \end{cases}$$

Remark: this result shows, after raising indices, that the trivial cobordism is the identity morphism of the Hilbert space.

Proof: Again,the proof goes through the computation of the connected invariants, and the equivalent statement in this case is:

$$A_d^\bullet(0|0,0)_{\eta,\mu} \quad = \quad \begin{cases} \frac{1}{d(s_1 s_2)} & \text{if } \eta = \mu = (d) \\ \\ 0 & \text{otherwise.} \end{cases}$$

As in the previous paragraph, we are concerned with integrals of the following type:

$$\int_{\overline{Adm}^\bullet_{h\xrightarrow{d}\mathbb{P}^1,(\eta 0,\mu\infty)}} c_{b_1}(\mathbb{E}^*)c_{b_2}(\mathbb{E}^*),$$

where $b_1 + b_2 = dim(\overline{Adm}^\bullet_{h\xrightarrow{d}\mathbb{P}^1,(\eta 0,\mu\infty)}) = 2h + \ell(\eta) + \ell(\mu) - 2$. Since $b_1 + b_2 \leq 2h$, we are forced to choose $\ell(\eta) = \ell(\mu) = 1$, i.e., full ramification over the points 0 and ∞.

By a relation of Mumford,

$$c_h(\mathbb{E}^*)^2 = 0,$$

the integral vanishes unless $h = 0$.

The nonvanishing integrals then are:

$$A_d^\bullet(0|0,0)_{(d),(d)} = \frac{1}{s_1 s_2}\int_{\overline{Adm}_{0\xrightarrow{d}\mathbb{P}^1,((d)0,(d)\infty)}} 1 \quad = \frac{1}{s_1 s_2}\frac{1}{d}.$$

4.5.3 The Level $(0,0)$ Pair of Pants

The invariants $A^\bullet_d(0|0,0)_{\eta,\nu,\mu}$ of the level $(0,0)$ pair of pants are computed by the integrals:

$$\int_{\overline{Adm}^\bullet_{h\stackrel{d}{\to}\mathbb{P}^1,(\eta 0,\mu 1,\nu\infty)}} c^{eq}_{2h-2}(-R^\bullet\pi_*f^*(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1})).$$

The dimension of the moduli space in question is

$$2h-d-2+\ell(\eta)+\ell(\mu)+\ell(\nu).$$

Hence, if $\ell(\eta)+\ell(\mu)+\ell(\nu) > d+2$, the relative connected integrals vanish. The disconnected integrals are then obtained inductively from invariants of lower degree d.

All other invariants have contributions from connected components, and hence need to be computed directly.

In an appendix to [BP04], Bryan, Pandharipande and Faber show that all invariants can be recursively determined from $A_d(0|0,0)_{(d),(d),(2)}$, the invariant corresponding to full ramification over two points, and a simple transposition over the third point. Their proof uses only TQFT formalism, and hence applies to our situation.

Lemma 6 *For d≥ 2,*

$$A_d(0|0,0)_{(d),(d),(2)} = \frac{1}{2}\frac{s_1+s_2}{s_1s_2}\left(d\cot\left(\frac{du}{2}\right)-\cot\left(\frac{u}{2}\right)\right).$$

Note: The above result differs from the analogous one in [BP04] by a factor of $-i$, that reflects a different normalization in their generating function conventions, that we do not wish to adopt.

PROOF: Notice, first of all, that the full ramification conditions force our covers to be connected. In this case the connected and disconnected invariants coincide.

According to (4.4), we have:

$$A_d(0|0,0)_{(d),(d),(2)} = \sum_{b_1+b_2=0}^{\infty} u^{b_1+b_2}s_1^{h-1-b_1}s_2^{h-1-b_2}\int_{\overline{Adm}_{h\stackrel{d}{\to}\mathbb{P}^1,((d)0,(d)1,(2)\infty)}} c_{b_1}(\mathbb{E}^*)c_{b_2}(\mathbb{E}^*),$$

with b_1+b_2 equal to the dimension of the moduli space, which is

$$dim(\overline{Adm}_{h\stackrel{d}{\to}\mathbb{P}^1,((d)0,(d)1,(2)\infty)}) = 2h-1.$$

For a given value of h, the only nonvanishing terms in the above expression are given by:

- $b_1 = h, \quad b_2 = h-1;$

- $b_1 = h-1, \quad b_2 = h.$

Adding the two terms, we obtain

$$A_d^h(0|0,0)_{(d),(d),(2)} = \frac{s_1+s_2}{s_1 s_2} \int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,((d)0,(d)1,(2)\infty)}} -\lambda_h \lambda_{h-1}$$

and consequently, the generating function:

$$A_d(0|0,0)_{(d),(d),(2)} = \frac{s_1+s_2}{s_1 s_2} \sum_{h=0}^{\infty} u^{2h-1} \int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,((d)0,(d)1,(2)\infty)}} -\lambda_h \lambda_{h-1},$$

where λ_k denotes the k^{th} Chern class of the Hodge bundle $\mathbb{E}$.

Let us recall that we defined the λ classes on moduli spaces of admissible covers simply by pulling them back from the appropriate moduli spaces of stable curves. In particular observe the diagram:

$$\begin{array}{ccc} \overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,((d)0,(d)1,(2)\infty)} & \xrightarrow{\rho} & \overline{M}_{h,2} \\ & \searrow & \downarrow \pi \\ & & \overline{M}_h \end{array}$$

The map ρ is defined by marking on the admissible covers the unique preimages of the branch points 0 and 1. The Hodge bundle on $\overline{M}_h$ pulls back to the Hodge bundle on $\overline{M}_{h,2}$, hence we can think of the λ classes on the moduli space of admissible covers as pulled back from $\overline{M}_{h,2}$.

Denote by $H_d \subset M_{h,2}$ the locus of curves admitting a degree d map to $\mathbb{P}^1$ which is totally ramified at the marked points. Let

$$\overline{H}_d \subset \overline{M}_{h,2}$$

be the closure of H_d, consisting of possibly nodal curves admitting a degree d map to a tree of rational curves, fully ramified over the two marked points. The image of the map

$$\rho : \overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,((d)0,(d)1,(2)\infty)} \longrightarrow \overline{M}_{h,2}$$

is precisely $\overline{H}_d$, and ρ is a degree $2h$ map onto its image.

From this we conclude that

$$\int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,((d)0,(d)1,(2)\infty)}} -\lambda_h \lambda_{h-1} \quad = 2h \int_{[\overline{H}_d]} -\lambda_h \lambda_{h-1}.$$

This is exactly the integral computed in [BP04], pages 28-29, hence the result follows. This concludes the proof of theorem 5.

4.5.4 The Calabi-Yau Cap

Now we come to the task of computing the invariants corresponding to the Calabi-Yau caps.

First of all let us notice that we can obtain $A_d(0|-1,0)_\eta$ from $A_d(0|0,-1)_\eta$ by simply interchanging the roles of s_1 and s_2.

We will compute the invariants $A_d(0|0,-1)_\eta$ by first computing the corresponding connected invariants, and then obtaining the disconnected ones by exponentiation. We compute the invariants explicitly in degrees $1, 2$ and 3, and present a conjecture for arbitrary degree d. The following statement is a theorem for $d = 1, 2, 3$ and conjectural for $d \geq 4$.

Statement 1 *The Calabi-Yau invariants are given by the following formula:*

$$\boxed{A_d(0|0,-1)_\eta = (-)^{d-\ell(\eta)} \frac{\left(2\sin\left(\frac{u}{2}\right)\right)^d}{(s_1)^{\ell(\eta)}\mathfrak{z}(\eta)\prod 2\sin\left(\frac{\eta_i u}{2}\right)}.} \tag{4.12}$$

This result is equivalent to the following formula for the connected invariants:

$$\boxed{A^\bullet_d(0|0,-1)_\eta = \begin{cases} \dfrac{(-)^{d-1}}{s_1}\dfrac{1}{d}\dfrac{\left(2\sin\left(\frac{u}{2}\right)\right)^d}{2\sin\left(\frac{du}{2}\right)} & \textit{for} \quad \eta = (d) \\ 0 & \textit{otherwise.} \end{cases}} \tag{4.13}$$

The vanishing of the connected invariants for all partitions but (d) is again a dimension count. We are interested in the integrals

$$\int_{\overline{Adm}_{h\xrightarrow{d}\mathbb{P}^1,(\eta\infty)}} c^{eq}_{2h+d-1}(-R^\bullet\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))). \tag{4.14}$$

Since $R^0\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0$, the virtual bundle

$$-R^\bullet\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1))$$

is in fact a vector bundle of rank $h + d - 1$ (by Riemann-Roch). When we evaluate expression (4.14) using formula (4.4), we obtain a sum of terms of the following type:

$$\int_{\overline{Adm}_{h\xrightarrow{d}\mathbb{P}^1,(\eta\infty)}} c_{b_1}(\mathbb{E}^*) c_{b_2}(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1))),$$

where $b_1 + b_2 = dim(\overline{Adm}_{h\xrightarrow{d}\mathbb{P}^1,(\eta\infty)}) = 2h + d + \ell(\eta) - 2$. But we also have the constraint $b_1 + b_2 \leq 2h + d - 1$, hence the only possibly nonvanishing integrals occur when $\ell(\eta) = 1$, i.e. when $\eta = (d)$. The indices b_1 and b_2 are forced to be, respectively, h and $h+d-1$. Notice that the full ramification condition forces all covers to be connected; the fully ramified connected and disconnected invariants coincide, thus allowing us to drop the superscript "$\bullet$". Finally, our task is to evaluate the integrals:

$$A^h_d(0|0,-1)_{(d)} = \frac{1}{s_1}\int_{\overline{Adm}_{h\xrightarrow{d}\mathbb{P}^1,((d)\infty)}} c_h(\mathbb{E}^*)c_{h+d-1}(R^1\pi_*f^*(\mathcal{O}_{\mathbb{P}^1}(-1))) =$$

$$= \frac{1}{s_1}\int_{\overline{Adm}_{h\xrightarrow{d}\mathbb{P}^1,((d)\infty)}} c_{2h+d-1}(R^1\pi_*f^*(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1))), \tag{4.15}$$

and organize them in generating function as follows:

$$A_d(0|0,-1)_{(d)} := \frac{1}{s_1}\sum_{h=0}^{\infty} u^{2h+d-1}A^h_d(0|0,-1)_{(d)}.$$

4.5.4.1 Notational Adjustments

In order to keep the actual computations cleaner, we have made some strategic and notational choices that we are about to explain:

ordering points: we choose to carry out our localization computation on moduli spaces of admissible covers where all ramification is specified. This allows us to unravel the combinatorics of the fixed loci in a more natural way. If we define

$$I_d(h) := \int_{\overline{Adm}_{h\xrightarrow{d}\mathbb{P}^1,((d)\infty,t_1,\ldots,t_{2h+d-1})}} c_{2h+d-1}(R^1\pi_*f^*(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-1))),$$

then the simple relation holds:

$$A^h_d(0|0,0)_{(d)} = \frac{1}{s_1}\frac{I_d(h)}{(2h+d-1)!}.$$

Our generating function is:

$$A_d(0|0,0)_{(d)} := \frac{1}{s_1}\sum_{h=0}^{\infty} u^{2h+d-1}\frac{I_d(h)}{(2h+d-1)!}.$$

not fixing points: in the rich network of moduli spaces of admissible covers, we can require ramification conditions over some fixed points of the base in general position, which we have been doing all along. From a computational point of view, integrating a given cohomology class α over a moduli space of covers ramified with type η over the fixed point x of the base is equivalent to integrating the class $\alpha \cap ev_1^*(x)$ over a moduli space of admissible covers where the first marked point has ramification type η and it is free to move around the base curve. This may seem like an irrelevant detail, and for the current computations it actually is, but it can become extremely important. When dealing with integrals with more than two marked points, we are allowed to "crash" together two of the marked points. The moduli spaces with multiple fixed ramification over a point may become singular; however the cohomological class $ev_1^*(x) \cap ev_2^*(x)$ can still make sense and allow us to carry on localization. For this reason, we choose to substitute in our computations

$$\int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,((d)\infty,t_1,\ldots,t_{2h+d-1})}} c_{2h+d-1}(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)))$$

with

$$\int_{\overline{Adm}_{h \xrightarrow{d} \mathbb{P}^1,((d),t_1,\ldots,t_{2h+d-1})}} ev_{(d)}^*(\infty) \cap c_{2h+d-1}(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))).$$

torus action: in our localization computations, we will let a one-dimensional torus $\mathbb{C}^*$ act on our bundles $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$. This action is completely unrelated to the canonical action introduced in page 38. As a matter of fact, part of our computational strategy consists in comparing results obtained changing this action on the bundles. In order to stress this independence from s_1 and s_2, we denote the equivariant parameter by $\hbar$.

4.5.4.2 Degree 1

In degree 1 the result is extremely simple. First of all, h is forced to be 0. The one and only integral to be computed is:

$$\int_{\overline{Adm}_{0 \xrightarrow{1} \mathbb{P}^1,(1)}} ev_{(1)}^*(\infty) = 1.$$

Hence

$$A_1(0|0,-1)_{(1)} = \frac{1}{s_1} \ ,$$

as expected.

4.5.5 Proof of Calabi-Yau Cap in Degree 2

We now carry out the explicit computation of the integral

$$I_2(h) = \int_{\overline{Adm}_{h \xrightarrow{2} \mathbb{P}^1_{\mathbb{C}}, (t_1, t_2, \cdots, t_{2h+2})}} ev_1^*(\infty) \cap c_{2h+1}(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))),$$

for all genera, and express the result in generating function form:

$$\mathcal{I}_2(u) = \sum_{h=0}^{\infty} \left(\frac{I_2(h)}{2h+1!} \right) u^{2h+1}.$$

Our invariants are then obtained dividing by s_1:

$$A_2(0|0,-1)_{(2)} = \frac{\mathcal{I}_2(u)}{s_1}.$$

4.5.5.1 The Strategy

It is important to notice that, while the final result is independent of the choice of the lifting of the $\mathbb{C}^*$ action to the vector bundle $E = R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, the intermediate calculations are not. This is in fact the heart of our strategy. We choose two different specific linearizations with the twofold objective of:

- limiting a priori the number and the combinatorial complexity of the contributing fixed loci;
- obtaining, by equating the calculations with the two linearizations, a recursive formula for genus h integrals in term of lower genus data.

4.5.5.2 The Localization Set-up

We induce different linearizations on the bundle E by choosing different liftings of the $\mathbb{C}^*$ action on the bundles $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$. Recall that a linearization of a line bundle over $\mathbb{P}^1$ is determined by the weights of the fixed fiber representations.

Linearization A: We choose to linearize the two bundles as follows:

weight :	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathcal{O}_{\mathbb{P}^1}$	0	0

There is only one fixed locus $F_{h,\cdot}$ contributing to the localization integral, consisting of a cover of $\mathbb{P}^1$ fully ramified over 0 and ∞, and a genus h curve mapping with degree 2 to an unparametrized $\mathbb{P}^1$ sprouting from the point 0. Figure 4.1 illustrates a cover

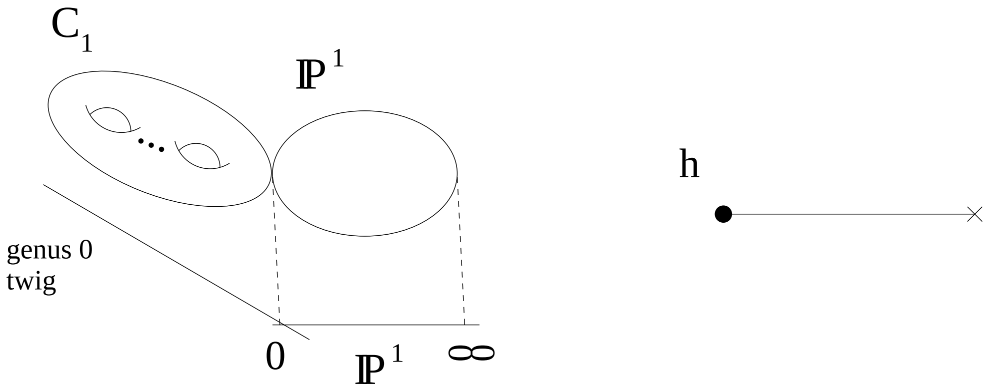


Figure 4.1. The fixed locus $F_{h,}$.

corresponding to a point in this fixed locus, and the conventional graph notation to indicate it.

The reason for this dramatic collapsing of the contributing fixed loci lies in some standard localization facts:

- the ramification condition required over ∞ implies that there can be only one connected component in the preimage of ∞. This translates to the fact that the localization graph can have at most 1 vertex over ∞;

- the weight 0 linearization of $\mathcal{O}_{\mathbb{P}^1}(-1)$ over ∞ implies that the localization graph must have valence 1 over ∞.

- finally, both bundles have weight 0 over ∞; the restriction of our bundle to fixed loci that have contracted components over ∞ involves the class $\lambda^2_{h_\infty}$, that vanishes for $h > 0$ by a famous result by Mumford ([Mum83]). The only option is then to have genus 0 over infinity. But a genus zero curve with only two special points is unstable, and hence must be contracted.

Linearization B: We choose to linearize the bundles with weights:

weight :	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathcal{O}_{\mathbb{P}^1}$	1	1

In this case the analysis of the possibly contributing fixed loci is similar, except we cannot appeal to Mumford's relation any more. Hence our fixed loci consist of a copy of

$\mathbb{P}^1$ ramified over 0 and ∞, with two curves of genus h_1,h_2 attached on either side (and, of course, $h_1+h_2=h$). These are the loci F_{h_1,h_2} described in Figure 2.1.

4.5.5.3 Explicit Evaluation of the Integral

LINEARIZATION A:

Let us first of all observe that $F_{h,\cdot}$ is naturally isomorphic to $\overline{Adm}_{h\xrightarrow{2}0,(t_1,t_2,\cdots,t_{2h+2})}$. Using the computations in chapter 2, and the standard equivariant cohomology fact that $ev_1^*(\infty)=-\hbar$, we obtain the explicit evaluation of our integral on this fixed locus:

$$I_2^A(h)=\int_{\overline{Adm}_{h\xrightarrow{2}0,(t_1,t_2,\cdots,t_{2h+2})}}\frac{\lambda_h\Lambda_h(1)(-\hbar/2)}{\hbar(\hbar-\psi)}=$$

$$-\frac{1}{2}\int_{\overline{Adm}_{h\xrightarrow{2}0,(t_1,t_2,\cdots,t_{2h+2})}}\lambda_h\lambda_{h-1}+\lambda_h\lambda_{h-2}\psi+\cdots+\lambda_h\psi^{h-1}.$$

Just as a convenient notation, let us denote the last integral by $L_2(h)$, so that

$$I_2^A(h)=-\frac{1}{2}L_2(h). \tag{4.16}$$

LINEARIZATION B:

In this case we have $h+1$ different types of fixed loci, corresponding to all possible ways of choosing an ordered pair of nonnegative integers adding to h. We will study separately three situations:

$F_{\cdot,h}$) This fixed locus is naturally isomorphic to $2h+1$ disjoint copies of $\overline{Adm}_{h\xrightarrow{2}0,(t_1,t_2,\cdots,t_{2h+2})}$. The evaluation of the integral reads:

$$\int_{F_{\cdot,h}}\frac{\lambda_h\Lambda_h(-1)(-\hbar/2)}{(-\hbar-\psi)}=$$

$$=-\frac{2h+1}{2}\int_{\overline{Adm}_{h\xrightarrow{2}0,(t_1,t_2,\cdots,t_{2h+2})}}\lambda_h\lambda_{h-1}+\lambda_h\lambda_{h-2}\psi+\cdots+\lambda_h\psi^{h-1}=-\frac{2h+1}{2}L_2(h).$$

F_{h_1,h_2}, $h_1,h_2\neq 0$) After keeping track of the combinatorics of the gluing and of the possible distributions of the marks, the integral evaluates:

$$\int_{F_{h_1,h_2}}\frac{\Lambda_{h_1}(-1)\Lambda_{h_1}(1)\lambda_{h_2}\Lambda_{h_2}(-1)(-\hbar/2)}{\hbar(\hbar-\psi)(-\hbar-\psi)}=$$

$$=-\binom{2h+1}{2h_2}\int_{\overline{Adm}_{h_1\xrightarrow{2}0}}(-)^{h_1}\psi^{2h_1-1}\int_{\overline{Adm}_{h_2\xrightarrow{2}0}}\lambda_{h_2}\lambda_{h_2-1}+\lambda_{h_2}\lambda_{h_2-2}\psi+\cdots+\lambda_{h_2}\psi^{h_2-1}=$$

$$:= (-)^{h_1+1}\binom{2h+1}{2h_2}P_2(h_1)L_2(h_2).$$

To make the notation lighter we omitted the marked points (that are still there though). Also, we choose to denote by P_2 the integral of ψ_t to the top power.

$F_{h,\cdot}$) This is the same fixed locus encountered in the computations with linearization A. However, the contribution in this case will be different:

$$\int_{F_{h,\cdot}} \frac{\Lambda_h(1)\Lambda_h(-1)(-\hbar/2)}{(\hbar-\psi)} =$$

$$= -\frac{1}{2}\int_{\overline{Adm}_{h\xrightarrow{2}0,(t_1,t_2,\cdots,t_{2h+2})}} (-)^h\psi^{2h-1} = (-)^{h+1}\frac{1}{2}P_2(h).$$

Allotgether, the integral computed with linearization B is:

$$I_2^B(h) = -\frac{1}{2}(2h+1)L_2(h) - \sum_{i=0}^{h-1}(-)^{h-i}\binom{2h+1}{2i}P_2(h-i)L(i),$$

where we have incorporated the last contribution in the summation by defining $L(0) = 1/2$.

Lemma 7 *For any i, $P(i) = \frac{1}{2}$.*

Proof: This follows easily from the fact that the ψ classes that we are using are pulled back on the space of admissible covers from $\overline{\mathcal{M}}_{0,2h+2}$ via an étale, degree 1/2 map (that accounts for the hyperelliptic involution upstairs). The projective coarse moduli space of $\overline{\mathcal{M}}_{0,2h+2}$ is $\overline{M}_{0,2h+2}$, and the two spaces are birational. It is a classical result that the integral of ψ to the top power on $\overline{M}_{0,2h+2}$ is one, hence the lemma.

We can now equate the results obtained with the two different linearizations, to obtain a recursive formula for the $L_2(h)$'s.

$$-\frac{1}{2}L_2(h) = -\frac{1}{2}(2h+1)L_2(h) - \sum_{i=0}^{h-1}(-)^{h-i}\binom{2h+1}{2i}P_2(h-i)L_2(i)$$

After a tiny bit of elementary arithmetic we obtain:

$$\boxed{L_2(h) = \frac{1}{2h}\sum_{i=0}^{h-1}(-)^{h-i+1}\binom{2h+1}{2i}L_2(i).} \tag{4.17}$$

4.5.5.4 The Generating Function

We now want to use relation (4.17) to compute the generating function:

$$\mathcal{L}_2(u) = \sum_{i=0}^{\infty} \left(\frac{L_2(i)}{2i+1!} \right) u^{2i+1}.$$

Let us first of all differentiate this function,

$$\frac{d}{du}\mathcal{L}_2(u) = \sum_{i=0}^{\infty} \left(\frac{L_2(i)}{2i!} \right) u^{2i}.$$

Now let us compute:

$$\frac{d}{du}\mathcal{L}_2(u) \cdot \sin(u) = \sum_{h=0}^{\infty} u^{2h+1} \sum_{i=0}^{h} (-)^{h-i+1} \frac{L_2(i)}{2i!(2h-2i+1)!} =$$

$$= \sum_{h=0}^{\infty} u^{2h+1} \sum_{i=0}^{h} (-)^{h-i+1} \binom{2h+1}{2i} \frac{L_2(i)}{2h+1!} = \sum_{h=0}^{\infty} u^{2h+1} \left(\frac{L_2(h)}{2h+1!} \right) = \mathcal{L}_2(u).$$

Hence relation (4.17) translates to the following ODE on the generating function $\mathcal{L}_2(u)$:

$$\boxed{\begin{array}{c} \mathcal{L}_2'(u) \cdot \sin(u) = \mathcal{L}_2(u), \\ \mathcal{L}_2(0) = 0. \end{array}} \tag{4.18}$$

This equation integrates to give us $\mathcal{L}_2(u) = \tan(u/2)$. Finally, recalling (4.16) we can conclude:

$$\boxed{\mathcal{I}_2(u) = -\frac{1}{2} \tan\left(\frac{u}{2}\right),} \tag{4.19}$$

which agrees (after one simple trig identity substitution) with the result in (4.13).

4.5.5.5 The Calabi-Yau Closed Sphere

Using result (4.19) it is now easy to compute via localization the connected invariants $A_2^\bullet(0|-1,-1)$ of the Calabi-Yau closed sphere. Consider the class of integrals

$$J_2(h) = \int_{\overline{Adm}_{h \xrightarrow{2} \mathbb{P}^1_{\mathbb{C}}, (t_1, t_2, \cdots, t_{2h+2})}} c_{2h+2}(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))).$$

Our invariants are then expressed by the generating function:

$$A_2^\bullet(0|-1,-1) = \sum_{h=0}^{\infty} \left(\frac{J_2(h)}{2h+2!} \right) u^{2h+2}.$$

Again, there is a particularly favorable choice of linearizations:

weight :	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathcal{O}_{\mathbb{P}^1}(-1)$	0	1

The only contributing fixed loci must have valence 1 both over 0 and ∞. These are precisely the loci F_{h_1,h_2} studied above. The explicit computation of the integral is:

$$: \quad (2h+2)\int \frac{\lambda_h \Lambda_h(1)\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)}{\hbar(\hbar-\psi)(-\hbar)} \quad = \quad \frac{1}{4}(2h+2)L_2(h)$$

$$: \quad 2\binom{2h+2}{2h_1+1}\int \frac{\lambda_{h_1}\Lambda_{h_1}(1)\left(\frac{\hbar}{2}\right)}{\hbar(\hbar-\psi)}\int \frac{\lambda_{h_2}\Lambda_{h_2}(-1)\left(-\frac{\hbar}{2}\right)}{-\hbar(-\hbar-\psi)} \quad = \quad \frac{1}{2}\binom{2h+2}{2h_1+1}L_2(h_1)L_2(h_2)$$

$$: \quad (2h+2)\int \frac{\lambda_h \Lambda_h(-1)\left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right)}{\hbar(-\hbar-\psi)(-\hbar)} \quad = \quad \frac{1}{4}(2h+2)L_2(h)$$

All previous integrals are computed over the appropriate unparmetrized admissible cover spaces. Adding everything together we obtain the relation:

$$J_2(h) = \frac{1}{2}\sum_{0}^{h}\binom{2h+2}{2i+1}L_2(i)L_2(h-i). \tag{4.20}$$

(Recalling that we have defined $L_2(0) = 1/2$.)

This relation allows us to obtain the generating function $A_2^{\bullet}(0|-1,-1)$. For this purpose it suffices to notice:

$$\boxed{A_2^{\bullet}(0|-1,-1) = 2\mathcal{I}_2(u)^2 = \frac{1}{2}\tan^2\left(\frac{u}{2}\right).} \tag{4.21}$$

Notice that we could compute these invariants also by gluing together two Calabi-Yau caps. The answer is, as it should be, exactly the same:

$$A_2^{\bullet}(0|-1,-1) = A_2(0|0,-1)_{(d)}A_2(0|-1,0)^{(d)} =$$

$$= 2s_1 s_2 A_2(0|0,-1)_{(d)}A_2(0|-1,0)_{(d)} =$$

$$= 2s_1 s_2\left(-\frac{1}{2s_1}\tan\left(\frac{u}{2}\right)\right)\left(-\frac{1}{2s_2}\tan\left(\frac{u}{2}\right)\right) = \frac{1}{2}\tan^2\left(\frac{u}{2}\right).$$

4.5.6 Proof of Calabi-Yau Cap in Degree 3

In this section we compute the integral

$$I_3(h) := \int_{\overline{Adm}_{h\xrightarrow{3}\mathbb{P}^1_{\mathbb{C}},((3),t_1,\cdots,t_{2h+2})}} ev^*_{(3)}(\infty) \cap c_{2h+2}(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))),$$

for all genera h, and present the result in generating function form:

$$\mathcal{I}_3(u) := \sum_{h=0}^{\infty} \frac{I_3(h)}{2h+2!} u^{2h+2}.$$

Our invariants are then obtained dividing by the equivariant parameter:

$$A_3(0|0,-1)_{(3)} = \frac{\mathcal{I}_3(u)}{s_1}.$$

4.5.6.1 The Strategy

We use localization to compute our integral. First of all, we choose an extremely convenient choice of linearizations on the $\mathbb{P}^1$-bundles $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$. This expresses our integral in terms of a Hodge integral over only one boundary component of the moduli space.

We then introduce an auxiliary integral, that we know to vanish for elementary dimension considerations. Evaluating this integral via localization produces relations between the integrals $I_3(h)$, for different genera h, integrals in degree 2 and simple Hurwitz numbers.

We are able to transform these relations into a linear differential equation for the generating function $\mathcal{I}_3(u)$. Finally, solving the ODE with the appropriate boundary conditions gives us the result.

4.5.6.2 The Localization Set-up

We choose to linearize the bundle as follows:

weight :	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathcal{O}_{\mathbb{P}^1}$	0	0

For completely analogous reasons to the degree two case (page 57), there is only one fixed locus, $F_{h,\cdot}$, contributing to the localization integral, consisting in a cover of $\mathbb{P}^1$ fully ramified over 0 and ∞, and a genus h curve mapping with degree 3 to an unparametrized $\mathbb{P}^1$ sprouting from the point 0.

The integral then becomes:

$$I_3(h) = \int_{F_{h,\cdot}} \frac{\lambda_h \Lambda_h(1)\frac{2}{9}\hbar^2}{\hbar(\hbar - \psi_3)} = \frac{2}{9}\int_{\overline{Adm}_{h\xrightarrow{3}0,((3),t_1,\cdots,t_{2h+2})}} \lambda_h\lambda_{h-1}\psi_3 + \lambda_h\lambda_{h-2}\psi_3^2 + \cdots + \lambda_h\psi_3^h.$$

With the sole purpose of keeping track of coefficients in a more natural way in what follows, we give a name to the rightmost integral without the 2/9 in front of it:

$$L_3(h) := \int_{\overline{Adm}_{h\xrightarrow{3}0,((3),t_1,\cdots,t_{2h+2})}} \lambda_h\lambda_{h-1}\psi_3 + \lambda_h\lambda_{h-2}\psi_3^2 + \cdots + \lambda_h\psi_3^h.$$

4.5.6.3 The Auxiliary Integral

Let us now consider the following equivariant integral:

$$\int_{\overline{Adm}_{h\xrightarrow{3}\mathbb{P}^1_{\mathbb{C}},(t_1,\cdots,t_{2h+4})}} ev_1^*(\infty) \cap c_{2h+2}(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))).$$

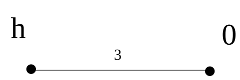
This integral must vanish for dimension reasons. Let us now evaluate this integral via localization.

We choose a different linearization for the trivial bundle:

weight :	over 0	over ∞
$\mathcal{O}_{\mathbb{P}^1}(-1)$	-1	0
$\mathcal{O}_{\mathbb{P}^1}$	1	1

With this choice of linearizations, the explicit evaluation of the integral follows. We are again invoking a famous relation by Mumford ([Mum83]):

$$\Lambda_h(-1)\Lambda_h(1) = (-)^h\hbar^{2h}. \tag{4.22}$$

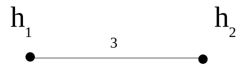
$F_{h,0})$ 

:

$$3\binom{2h+3}{2h+2}\int_{\overline{Adm}_{h\xrightarrow{3}0,((3),t_1,\cdots,t_{2h+2})}} \frac{(-)^h\hbar^{2h}}{\hbar(\hbar-\psi_3)} \quad \int_{\overline{Adm}_{0\xrightarrow{3}0,((3),t_1,t_2)}} \frac{1}{-\hbar-\psi_3}\left(\frac{2}{9}\hbar^2\right) =$$

$$= (-)^{h+1}\frac{2}{3}\binom{2h+3}{2h+2}\frac{1}{\hbar}\int_{\overline{Adm}_{h\xrightarrow{3}0,((3),t_1,\cdots,t_{2h+2})}} \psi^{2h} \quad \int_{\overline{Adm}_{0\xrightarrow{3}0,((3),t_1,t_2)}} 1 =$$

$$= \boxed{(-)^{h+1}\frac{2}{3}\binom{2h+3}{2h+2}P_{3,(3)}(h)L_3(0)\frac{1}{\hbar}.}$$

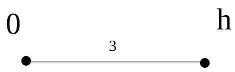
$F_{h_1,h_2})$ 

:

$$3\binom{2h+3}{2h_1+2}\int_{\overline{Adm}_{h_1\xrightarrow{3}0,((3),t_1,\cdots,t_{2h_1+2})}}\frac{(-)^{h_1}\hbar^{2h_1}}{\hbar(\hbar-\psi_3)}\quad\int_{\overline{Adm}_{h_2\xrightarrow{3}0,((3),t_1,\cdots,t_{2h_2+2})}}\frac{\lambda_{h_2}\Lambda_{h_2}(-1)}{-\hbar-\psi_3}\left(\frac{2}{9}\hbar^2\right)=$$

$$=(-)^{h_1+1}\frac{2}{3}\binom{2h+3}{2h_1+2}\frac{1}{\hbar}\int_{\overline{Adm}_{h_1\xrightarrow{3}0,((3),t_1,\cdots,t_{2h_1+2})}}\psi^{2h_1}\int_{\overline{Adm}_{h_2\xrightarrow{3}0,((3),t_1,\cdots,t_{2h_2+2})}}\lambda_{h_2}\lambda_{h_2-1}\psi_3+\cdots+\lambda_{h_2}\psi_3^{h_2}=$$

$$=\boxed{(-)^{h_1+1}\frac{2}{3}\binom{2h+3}{2h_1+2}P_{3,(3)}(h_1)L_3(h_2)\frac{1}{\hbar}.}$$

$F_{0,h})$ 

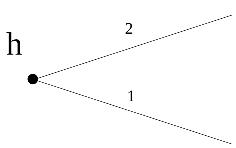
:

$$3\binom{2h+3}{2}\int_{\overline{Adm}_{0\xrightarrow{3}0,((3),t_1,t_2)}}\frac{1}{\hbar(\hbar-\psi_3)}\quad\int_{\overline{Adm}_{h\xrightarrow{3}0,((3),t_1,\cdots,t_{2h+2})}}\frac{\lambda_h\Lambda_h(-1)}{-\hbar-\psi_3}\left(\frac{2}{9}\hbar^2\right)=$$

$$=(-)^{h+1}\frac{2}{3}\binom{2h+3}{2}\frac{1}{\hbar}\int_{\overline{Adm}_{0\xrightarrow{3}0,((3),t_1,t_2)}}1\int_{\overline{Adm}_{h\xrightarrow{3}0,((3),t_1,\cdots,t_{2h+2})}}\lambda_h\lambda_{h-1}\psi_3+\cdots+\lambda_h\psi_3^h=$$

$$=\boxed{-\frac{2}{3}\binom{2h+3}{2}P_{3,(3)}(0)L_3(h)\frac{1}{\hbar}.}$$

$F_{h,\cdot,\cdot})$

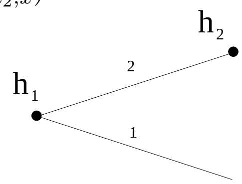


:

$$\binom{2h+3}{2h+3}\int_{\overline{Adm}_{h\xrightarrow{3}0,(t_1,\cdots,t_{2h+4})}}\frac{(-)^h\hbar^{2h}}{\hbar(\hbar-\psi_t)}\left(\frac{1}{2}\hbar^2\right)=$$

$$=(-)^h\frac{1}{2}\frac{1}{\hbar}\int_{\overline{Adm}_{h\xrightarrow{3}0,(t_1,\cdots,t_{2h+4})}}\psi_t^{2h+1}\quad=\quad\boxed{(-)^h\frac{1}{2}P_{3,(t)}(h)\frac{1}{\hbar}.}$$

$F_{h_1,h_2,x})$



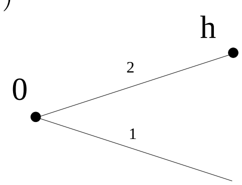
:

$$2\binom{2h+3}{2h_1+3}\int_{\overline{Adm}_{h_1\xrightarrow{3}0,(t_1,\cdots,t_{2h_2+4})}}\frac{(-)^{h_1}\hbar^{2h_1}}{\hbar(\hbar-\psi_t)}\int_{\overline{Adm}_{h_2\xrightarrow{2}0,(t_1,\cdots,t_{2h_2+2})}}\frac{\lambda_{h_2}\Lambda_{h_2}(-1)}{-\hbar-\psi_t}\left(\frac{1}{2}\hbar^2\right)=$$

$$=(-)^{h_1}\binom{2h+3}{2h_1+3}\frac{1}{\hbar}\int_{\overline{Adm}_{h_1\xrightarrow{3}0,(t_1,\cdots,t_{2h_2+4})}}\psi_t^{2h_1+1}\int_{\overline{Adm}_{h_2\xrightarrow{2}0,(t_1,\cdots,t_{2h_2+2})}}\lambda_{h_2}\lambda_{h_2-1}+\cdots+\lambda_{h_2}\psi_t^{h-1}=$$

$$=\boxed{(-)^{h_1}\binom{2h+3}{2h_1+3}P_{3,(t)}(h_1)L_2(h_2)\frac{1}{\hbar}.}$$

$F_{0,h,\cdot})$



:

$$2\binom{2h+3}{3}\int_{\overline{Adm}_{0\xrightarrow{3}0,(t_1,\cdots,t_4)}}\frac{1}{\hbar(\hbar-\psi_t)}\int_{\overline{Adm}_{h\xrightarrow{2}0,(t_1,\cdots,t_{2h+2})}}\frac{\lambda_h\Lambda_h(-1)}{-\hbar-\psi_t}\left(\frac{1}{2}\hbar^2\right)=$$

$$=\binom{2h+3}{3}\frac{1}{\hbar}\int_{\overline{Adm}_{0\xrightarrow{3}0,(t_1,\cdots,t_4)}}\psi_t\int_{\overline{Adm}_{h\xrightarrow{2}0,(t_1,\cdots,t_{2h+2})}}\lambda_h\lambda_{h-1}+\cdots+\lambda_h\psi_t^{h-1}=$$

$$=\boxed{\binom{2h+3}{3}P_{3,(t)}(0)L_2(h)\frac{1}{\hbar}.}$$

Finally, adding everything up, we obtain the following relation:

$$\boxed{0=\frac{2}{3}\sum_{i=0}^{h}\binom{2h+3}{2i+1}(-)^{h-i+1}P_{3,(3)}(h-i)L_3(i)+\sum_{i=0}^{h}\binom{2h+3}{2i}(-)^{h-i}P_{3,(t)}(h-i)L_2(i).}\tag{4.23}$$

4.5.6.4 The Generating Function

Now for the less deep but more delicate part of our computation: we need to extract from relation (4.23) a differential equation involving our generating function.

Let us start with a preliminary lemma:

Lemma 8 *For all $h \geq 0$:*

1. $P_{3,(3)}(h) = 3^{2h}$.

2. $P_{3,(t)}(h) = (3^{2h+2} - 1)/2$.

PROOF: 1. Consider the map

$$\begin{array}{c} \overline{Adm}_{h \xrightarrow{3} 0,((3),t_1,\cdots,t_{2h+2})} \\ \downarrow \pi \\ \overline{M}_{0,2h+3}. \end{array}$$

It is a classical result that

$$\int_{\overline{M}_{0,2h+3}} \psi_1^{2h} = 1.$$

Since our psi class is just the pull-back of ψ_1 on $\overline{\mathcal{M}}_{0,2h+3}$, and this space is birational to its projective coarse moduli space $\overline{M}_{0,2h+3}$, our lemma is proven if we show that π has degree 3^{2h}. This is a classic Hurwitz number, counting the number of degree 3 covers of the Riemann sphere with a triple ramification point and simple ramification otherwise. The problem is purely combinatorial. We are free to choose a three-cycle in S_3 giving the monodromy of the triple point. The triple point automatically guarantees that our cover is connected. Then we are free to choose cycles for the first $(2h+1)$ simple ramification points. The monodromy of the last ramification point is determined by the fact that the product of all monodromies should be the identity. So all together we had a choice of $2 \cdot 3^{2h+1}$ elements of S_3. We now need to divide by the conjugation action of S_3 on itself, that geometrically amounts to simply relabelling the sheets of the cover. Finally we obtain the desired 3^{2h} non isomorphic covers.

2. Similarly, we need to count the number of degree 3 covers of $\mathbb{P}^1$ with $2h+4$ simple ramification points. Paralleling the previous argument, we can choose $(2h+3)$ cycles freely. But we have to beware of disconnected covers. These can happen only if we chose always the same cycle. So in total we have $3^{2h+3} - 3$ choices. Dividing now by 6 we obtain our claim.

Let us now translate relation (4.23) in the language of generating functions. Define:

- $\mathcal{L}_3(u) := \sum_{h=0}^{\infty} \frac{L_3(h)}{2h+2!} u^{2h+2}$.
- $\mathcal{L}_2(u) := \sum_{h=0}^{\infty} \frac{L_2(h)}{2h+1!} u^{2h+1}$.
- $\mathcal{P}_{3,(3)}(u) := \sum_{h=0}^{\infty} (-)^h \frac{P_{3,(3)}(h)}{2h+2!} u^{2h+2}$.
- $\mathcal{P}_{3,(t)}(u) := \sum_{h=0}^{\infty} (-)^h \frac{P_{3,(t)}(h)}{2h+3!} u^{2h+3}$.

Then our relation (4.23) becomes an ordinary differential equation on the generating functions:

$$\boxed{\frac{2}{3}\mathcal{P}_{3,(3)}\mathcal{L}_3' - \mathcal{P}_{3,(t)}\mathcal{L}_2' = 0.} \tag{4.24}$$

By lemma 8 we can explicitly describe the Hurwitz numbers' generating functions:

$$\mathcal{P}_{3,(3)}(u) = \frac{1-\cos(3u)}{9};$$

$$\mathcal{P}_{3,(t)}(u) = \frac{3\sin(u) - \sin(3u)}{6}.$$

Also, we do know the generating function for the degree 2 theory, hence:

$$\mathcal{L}_2'(u) = \frac{d}{du}\tan\left(\frac{u}{2}\right) = \frac{1}{2\cos^2\left(\frac{u}{2}\right)}.$$

Finally, we have reduced our problem to integrating the following:

$$\boxed{\begin{array}{c} \tilde{\mathcal{L}}_3'(u) = \dfrac{9}{8}\dfrac{3\sin(u) - \sin(3u)}{(1-\cos(3u))\cos^2\left(\frac{u}{2}\right)}, \\ \tilde{\mathcal{L}}_3(0) = 0. \end{array}} \tag{4.25}$$

This ODE integrates to

$$\mathcal{L}_3(u) = \frac{9}{2}\left(\frac{1}{4\cos^2\left(\frac{u}{2}\right) - 1} - \frac{1}{3}\right).$$

Now let us remember that the generating function $\mathcal{I}_3(u)$ is simply $(2/9)\mathcal{L}_3(u)$. After just a little bit of trigonometry clean-up we obtain:

$$\boxed{\mathcal{I}_3(u) = \frac{4}{3}\frac{\sin^3\left(\frac{u}{2}\right)}{\sin\left(\frac{3u}{2}\right)}} \tag{4.26}$$

4.5.6.5 The Calabi-Yau Closed Sphere

In a completely similar fashion to degree 2, it is possible to obtain from the previous computation the connected invariants for the Calabi-Yau closed sphere. For a given genus h these reduce to:

$$J_3(h) = \int_{\overline{Adm}_{h \xrightarrow{3} \mathbb{P}^1_{\mathbb{C}}, (t_1, \cdots, t_{2h+4})}} c_{2h+4}(R^1\pi_* f^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))).$$

The invariants are:

$$\boxed{A_3^{\bullet}(0| -1, -1) = \sum_{h=0}^{\infty} \left(\frac{J_3(h)}{2h+4!} \right) u^{2h+4} = 3\mathcal{I}_3(u)^2 = \frac{16}{3} \frac{\sin^6\left(\frac{u}{2}\right)}{\sin^2\left(\frac{3u}{2}\right)}.} \tag{4.27}$$

The result can be obtained either with a direct localization computation or by gluing two Calabi-Yau caps.

4.6 Specializations of the Theory

We now discuss two particular specializations of the theory, obtained by embedding a one-dimensional torus inside the two-dimensional torus T, and considering the theory as depending from one equivariant parameter instead of two.

When we specialize to the anti-diagonal action, we notice that the coefficients for the product simplify dramatically. It is possible to obtain nice closed formulas for our theory, and to view our TQFT as a one parameter deformation of the classical TQFT of Hurwitz numbers studied by Dijkgraaf and Witten in [DW90]. Our formulas show connections to the representation theory of the symmetric group S_d. The relevant representation theoretic quantities are introduced in Appendix A.

The diagonal specialization is how Bryan and Pandharipande's theory was actually originally born, in [BP03]. Here the coefficients of the embedded Calabi-Yau TQFT (where $a + b = -1$, using the notation of section 3.5) are related to the Gopakumar-Vafa invariants for a local curve. Although we know that this TQFT is semisimple, finding reasonable closed formulas is quite a hard task, and so far has only been accomplished in degrees 2 and 3.

4.6.1 The Anti-diagonal Action

Let $\mathbb{C}^*$ be embedded in the two-dimensional torus T via the map

$$\alpha \mapsto \left(\alpha, \frac{1}{\alpha} \right).$$

$\mathbb{C}^*$ acts on N by composing this embedding with the natural action of T constructed in page 38. If we let

$$H^*_{\mathbb{C}^*}(pt) = \mathbb{C}[s],$$

then the one parameter theory obtained with this action corresponds to setting

$$s = s_1 = -s_2.$$

4.6.1.1 The Q-dimension of an Irreducible Representation

Let ρ be an irreducible representation of the symmetric group on d letters S_d. Classically, a Young diagram, and hence a partition of d, can be canonically associated to ρ (see Appendix A).

We now define the Q-dimension of the representation ρ to be:

$$\frac{dim_Q \rho}{d!} := \prod_{\square \in \rho} \frac{1-Q}{1-Q^{h(\square)}} = \prod_{\square \in \rho} \frac{1}{1+Q+\cdots+Q^{h(\square)-1}} \tag{4.28}$$

Formula (4.28) specializes to the ordinary dimension of ρ when setting $Q = 1$. It is a consequence of the classical hooklenght formula (A.1).

4.6.1.2 The Level $(0,0)$ TQFT

The main result is that the level $(0,0)$ TQFT completely collapses to the Dijkgraaf TQFT $\mathcal{D}$. In particular, we have explicit formulas for the semisimple basis of the Frobenius algebra. The basis vectors are indexed by irreducible representations of the symmetric group S_d.

Lemma 9 (Bryan-Pandharipande) *For the anti-diagonal action, the level* $(0,0)$ *series have no nonzero terms of positive degree in* u.

PROOF: endow $\mathbb{C}$ with the $\mathbb{C}^*$ action

$$\alpha \cdot z = \alpha^n z.$$

You can regard it as an equivariant line bundle over a point, whose first equivariant Chern class is ns. Let us denote such equivariant line bundle by $\mathbb{C}_{ns}$.

The level $(0,0)$ partition functions are, up to some pure weight factor, constructed from integrals of the form:

$$\int_{\overline{Adm}_{h \xrightarrow{d} X,(\eta^1 x_1,\ldots,\eta^r x_r)}} e^{eq}(\mathbb{E}^* \otimes \mathbb{C}_s) e^{eq}(\mathbb{E}^* \otimes \mathbb{C}_{-s}) =$$

$$\int_{\overline{Adm}_{h \xrightarrow{d} X,(\eta^1 x_1,\ldots,\eta^r x_r)}} (-)^h e^{eq}((\mathbb{E}^* \oplus \mathbb{E}) \otimes \mathbb{C}_s).$$

Equivariant Chern classes of a bundle also are products of ordinary Chern classes times the appropriate factor of s. But by Mumford's relation (4.22), all Chern classes (but the 0-th) of the bundle $\mathbb{E}^* \oplus \mathbb{E}$ vanish. Hence the only possibly nonvanishing integrals occur when the dimension of the moduli space is 0, which then constitutes the degree 0 term in our generating functions.

We have therefore already essentially produced a semisimple basis for the corresponding Frobenius algebra in page 34. All we need to do is to adjust for the equivariant parameter:

semisimple basis: let ρ be an irreducible representation of the symmetric group S_d, χ_ρ its character function; a semisimple basis for the level $(0,0)$ TQFT is given by the vectors

$$\mathrm{e}_\rho = \frac{dim\rho}{d!} \sum_{\eta \vdash d} (s)^{\ell(\eta)-d} \chi_\rho(\eta) \mathrm{e}_\eta. \tag{4.29}$$

4.6.1.3 The Structure of the Weighted TQFT

Theorem 6 *The closed partition functions for the weighted TQFT are given by the following closed formulas:*

$$\boxed{A_d(g|k_1,k_2) = (-1)^{d(g-1-k_2)} s^{d(2g-2-k_1-k_2)} \sum_\rho \left(\frac{d!}{dim\rho}\right)^{2g-2} \left(\frac{dim\rho}{dim_Q\rho}\right)^{k_1+k_2} Q^{n(\rho)k_1+n(\rho')k_2},} \tag{4.30}$$

where:

- *the variable* $Q = e^{iu}$

- *ρ denotes an irreducible representation of the symmetric group S_d;*
- *ρ' denotes the dual representation;*
- *the function n is defined in section A.2.*

Remark. Notice that by setting $Q = 1$, which corresponds to $u = 0$, we recover the classical formula (3.4) counting unramified covers of a genus g topological surface. Thus any TQFT naturally embedded in our weighted TQFT constitutes a one parameter deformation of the Dijkgraaf TQFT.

PROOF: by theorem 4, to completely describe the structure of a semisimple weighted TQFT it suffices to evaluate the following quantities:

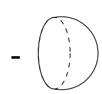
λ_ρ**:** the e_ρ-eigenvalue of the genus adding operator, or, equivalently, the inverse of the counit evaluated on e_ρ;

μ_ρ**:** the e_ρ-eigenvalue of the left level-subtracting operator, or, equivalently, the coefficient of e_ρ in the $(0,-1)$ Calabi-Yau cap;

$\overline{\mu}_\rho$**:** the e_ρ-eigenvalue of the right level-subtracting operator, or, equivalently, the coefficient of e_ρ in the $(-1,0)$ Calabi-Yau cap.

The computation of λ_ρ coincides exactly with Bryan and Pandharipande's computation in [BP04]. We reproduce it here for the sake of completeness.

$$\lambda_\rho^{-1} = \mathcal{U}\left(\quad \right)(\mathrm{e}_\rho)$$



(0, 0)

$$= \frac{dim\rho}{d!}\sum_{\eta}(is)^{\ell(\eta)-d}\chi_\rho(\eta)A_d(0|0,0)_\eta$$

$$= \frac{dim\rho}{d!}(is)^{\ell(1^d)-d}\chi_\rho(1^d)\frac{1}{d!(-s^2)^d}$$

$$= \left(\frac{dim\rho}{d!}\right)^2(is)^{-2d}$$

Hence,

$$\lambda_\rho = \left(\frac{d!}{dim\rho}\right)^2(is)^{2d}$$

To compute μ_ρ and $\overline{\mu}_\rho$ let us first of all observe that the tensors associated to the Calabi-Yau caps in our theory are scalar multiples of the tensors in Bryan and Pandharipande's theory.

$$\mathcal{U}(CY\,cap) = 2^d \sin\left(\frac{u}{2}\right)^d \mathcal{BP}(CY\,cap) = \frac{(1-Q)^d}{Q^{\frac{d}{2}}(-i)^d}\mathcal{BP}(CY\,cap).$$

This observation, together with the formulas in [BP04], page 36, implies:

$$\mu_\rho = s^d \frac{d!}{dim\rho}(1-Q)^d s_\rho(Q),$$

$$\overline{\mu}_\rho = (-s)^d \frac{d!}{dim\rho}(Q)(1-Q)^d s_{\rho'},$$

where s_ρ denotes the Schur function of the representation ρ, and is defined to be ([Mac95]):

$$s_\rho(Q) := Q^{n(\rho)} \prod_{\square\in\rho} \frac{1}{1-Q^{h(\square)}}.$$

Plugging this in, we obtain:

$$\begin{aligned}
\mu_\rho &= s^d \left(\frac{d!}{dim\rho}\right)(1-Q)^d Q^{n(\rho)} \prod_{\square\in\rho} \frac{1}{1-Q^{h(\square)}} \\
&= s^d \left(\frac{d!}{dim\rho}\right) Q^{n(\rho)} \prod_{\square\in\rho} \frac{1-Q}{1-Q^{h(\square)}} \\
&= s^d \left(\frac{dim_Q\rho}{dim\rho}\right) Q^{n(\rho)}
\end{aligned}$$

and

$$\begin{aligned}
\overline{\mu}_\rho &= (-s)^d \left(\frac{d!}{dim\rho}\right)(1-Q)^d Q^{n(\rho')} \prod_{\square\in\rho'} \frac{1}{1-Q^{h(\square)}} \\
&= s^d \left(\frac{d!}{dim\rho}\right) Q^{n(\rho')} \prod_{\square\in\rho'} \frac{1-Q}{1-Q^{h(\square)}} \\
&= s^d \left(\frac{dim_Q\rho}{dim\rho}\right) Q^{n(\rho')}
\end{aligned}$$

Theorem 6 is obtained by simply using these coefficients in the formula given by theorem 4.

4.6.2 The Diagonal Action

Let $\mathbb{C}^*$ be embedded in the two dimensional torus T via the diagonal map

$$\alpha \mapsto (\alpha, \alpha)\,.$$

$\mathbb{C}^*$ acts on N by composing this embedding with the natural action of T constructed in page 38. If we let

$$H^*_{\mathbb{C}^*}(pt) = \mathbb{C}[s],$$

then considering the one level theory obtained with this action corresponds to setting

$$s = s_1 = s_2.$$

In the case of the diagonal action, the invariants depend only on the sum of the levels. We, therefore obtain a one-parameter weighted TQFT.

We choose to adopt the following notation:

$$A_d(g|k_1, k_2)_{\eta_1,\ldots,\eta_n}{}_{|s_1=s_2=s} := A_d^{\Delta}(g|k_1 + k_2)_{\eta_1,\ldots,\eta_n} \tag{4.31}$$

4.6.2.1 Degree 2

It is possible to diagonalize and express closed formulas for the general theory in degree two. In the case of the diagonal action, these formulas are particularly elegant and simple.

Unfortunately, we do not have a semisimple basis readily at hand. However, we can still "diagonalize" the theory by computing the eigenvalues of the genus adding and level-subtracting operators. Let us begin by constructing these operators out of our known information. Recall:

Calabi-Yau cap (unit):

$$\begin{aligned} A_2^{\Delta}(0|-1)^{(1^2)} &= s^2 \\ A_2^{\Delta}(0|-1)^{(2)} &= -s\tan\left(\tfrac{u}{2}\right) \end{aligned}$$

Level 0 pair of pants (multiplication):

$$A_2^\Delta(0|0)^{(1^2)}_{(1^2),(1^2)} = 1 \qquad A_2^\Delta(0|0)^{(2)}_{(1^2),(1^2)} = 0$$

$$A_2^\Delta(0|0)^{(1^2)}_{(1^2),(2)} = 0 \qquad A_2^\Delta(0|0)^{(2)}_{(1^2),(2)} = 1$$

$$A_2^\Delta(0|0)^{(1^2)}_{(2),(1^2)} = 0 \qquad A_2^\Delta(0|0)^{(2)}_{(2),(1^2)} = 1$$

$$A_2^\Delta(0|0)^{(1^2)}_{(2),(2)} = s^2 \qquad A_2^\Delta(0|0)^{(2)}_{(2),(2)} = -2s\tan\left(\frac{u}{2}\right)$$

We obtain the level-subtracting operator by taking multiplication by the Calabi-Yau unit. Using tensor notation:

$$A_2^\Delta(0|-1)^\lambda_\eta = A_2^\Delta(0|-1)^\mu A_2^\Delta(0|0)^\lambda_{\mu,\eta}.$$

We organize the result in a matrix in the obvious way.

Level-subtracting operator:

$$A = \begin{bmatrix} s^2 & -s^3\tan\left(\frac{u}{2}\right) \\ -s\tan\left(\frac{u}{2}\right) & s^2\left(1+2\tan\left(\frac{u}{2}\right)\right) \end{bmatrix}$$

For the genus adding operator, we need to glue two pairs of pants. Again, in tensor notation:

$$A_2^\Delta(1|0)^\lambda_\eta = A_2^\Delta(0|0)^\lambda_{\mu,\nu} A_2^\Delta(0|0)^{\mu,\nu}_\eta,$$

yielding:

Genus-adding operator:

$$G = \begin{bmatrix} 4s^4 & -4s^5\tan\left(\frac{u}{2}\right) \\ -4s^3\tan\left(\frac{u}{2}\right) & 4s^4\left(1+2\tan\left(\frac{u}{2}\right)\right) \end{bmatrix} = 4s^2 A$$

The two matrices are scalar multiples of each other. We can easily extract the eigenvalues of A:

$$\lambda_1 = \frac{s^2}{1-\sin\left(\frac{u}{2}\right)},$$

$$\lambda_2 = \frac{s^2}{1+\sin\left(\frac{u}{2}\right)}.$$

Now, from theorem 4 we can obtain the partition functions for the closed invariants.

Theorem 7

$$A_2^{\Delta}(g|k) = s^{2(2g-2-k)}2^{2g-2}\left\{\left(1-\sin\left(\frac{u}{2}\right)\right)^{k+1-g}+\left(1+\sin\left(\frac{u}{2}\right)\right)^{k+1-g}\right\}$$

In particular, we obtain the following beautiful formula for the Calabi-Yau embedded TQFT:

$$\boxed{A_2^{\Delta}(g|2g-2) = 2^{2g-2}\left\{\left(1-\sin\left(\frac{u}{2}\right)\right)^{g-1}+\left(1+\sin\left(\frac{u}{2}\right)\right)^{g-1}\right\}}$$

4.6.2.2 Degree 3

In degree 3 things are significantly more complicated. It is not hard to compute the coefficients for the genus-adding and level-subtracting operators. However, their eigenvalues are not easily expressible in terms of our variables. Let us begin, again, by recalling the coefficients of our Calabi-Yau cap and level 0 pair of pants. We choose to express these coefficients in the variable

$$Q = e^{iu}.$$

Calabi-Yau cap (unit):

$$\begin{aligned} A_3^{\Delta}(0|-1)^{(1^3)} &= s^3 \\ A_3^{\Delta}(0|-1)^{(2)} &= is\tfrac{(Q-1)^2}{Q^2-1} \\ A_3^{\Delta}(0|-1)^{(3)} &= -s\tfrac{(Q-1)^3}{Q^3-1} \end{aligned}$$

Level 0 pair of pants (multiplication):

$$\begin{array}{llllll}
A_3^{\Delta}(0|0)^{(1^3)}_{(1^3),(1^3)} = 1 & A_3^{\Delta}(0|0)^{(2)}_{(1^3),(1^3)} = 0 & A_3^{\Delta}(0|0)^{(3)}_{(1^3),(1^3)} = 0 \\
A_3^{\Delta}(0|0)^{(1^3)}_{(1^3),(2)} = 0 & A_3^{\Delta}(0|0)^{(2)}_{(1^3),(2)} = 1 & A_3^{\Delta}(0|0)^{(3)}_{(1^3),(2)} = 0 \\
A_3^{\Delta}(0|0)^{(1^3)}_{(1^3),(3)} = 0 & A_3^{\Delta}(0|0)^{(2)}_{(1^3),(3)} = 0 & A_3^{\Delta}(0|0)^{(3)}_{(1^3),(3)} = 1 \\
A_3^{\Delta}(0|0)^{(1^3)}_{(2),(1^3)} = 0 & A_3^{\Delta}(0|0)^{(2)}_{(2),(1^3)} = 1 & A_3^{\Delta}(0|0)^{(3)}_{(2),(1^3)} = 0 \\
A_3^{\Delta}(0|0)^{(1^3)}_{(2),(2)} = 3s^2 & A_3^{\Delta}(0|0)^{(2)}_{(2),(2)} = 2is\frac{(Q-1)^2}{Q^2-1} & A_3^{\Delta}(0|0)^{(3)}_{(2),(2)} = 3 \\
A_3^{\Delta}(0|0)^{(1^3)}_{(2),(3)} = 0 & A_3^{\Delta}(0|0)^{(2)}_{(2),(3)} = 2s^2 & A_3^{\Delta}(0|0)^{(3)}_{(2),(3)} = 6is\frac{(Q-1)(Q^2-1)}{Q^3-1} \\
A_3^{\Delta}(0|0)^{(1^3)}_{(3),(1^3)} = 0 & A_3^{\Delta}(0|0)^{(2)}_{(3),(1^3)} = 0 & A_3^{\Delta}(0|0)^{(3)}_{(3),(1^3)} = 1 \\
A_3^{\Delta}(0|0)^{(1^3)}_{(3),(2)} = 0 & A_3^{\Delta}(0|0)^{(2)}_{(3),(2)} = 2s^2 & A_3^{\Delta}(0|0)^{(3)}_{(3),(2)} = 6is\frac{(Q-1)(Q^2-1)}{Q^3-1} \\
A_3^{\Delta}(0|0)^{(1^3)}_{(3),(3)} = 2s^4 & A_3^{\Delta}(0|0)^{(2)}_{(3),(3)} = 4is^3\frac{(Q-1)(Q^2-1)}{Q^3-1} &
\end{array}$$

$$A_3^{\Delta}(0|0)^{(3)}_{(3),(3)} = s^2\left(1-4\frac{(Q-1)^4(2(Q-1)^2+9)}{(Q^3-1)^2}\right)$$

By gluing appropriately, we can obtain our desired operators:

Genus-adding operator :

$$G = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix},$$

with:

- $g_{11} = 18s^6$;
- $g_{12} = \frac{12\,i\,(4\,Q^3+3\,Q^2-3\,Q-4)\,s^7}{(Q^2+Q+1)\,(Q+1)}$;
- $g_{13} = -\frac{6\,(5\,Q^4-2\,Q^3-33\,Q^2-2\,Q+5)\,s^8}{(Q^2+Q+1)^2}$;
- $g_{21} = \frac{4\,i\,(4\,Q^3+3\,Q^2-3\,Q-4)\,s^5}{(Q^2+Q+1)\,(Q+1)}$;
- $g_{22} = -\frac{4\,s^6\,(11\,Q^6-90\,Q^4-166\,Q^3-90\,Q^2+11)}{(Q^2+Q+1)^2\,(Q+1)^2}$;
- $g_{23} = \frac{-4\,i\,s^7\,(7\,Q^7-13\,Q^6-150\,Q^5-128\,Q^4+128\,Q^3+150\,Q^2+13\,Q-7)}{(Q^2+Q+1)^3\,(Q+1)}$;
- $g_{31} = -\frac{3\,(5\,Q^4-2\,Q^3-33\,Q^2-2\,Q+5)\,s^4}{(Q^2+Q+1)^2}$;
- $g_{32} = \frac{-6\,i\,s^5\,(7\,Q^7-13\,Q^6-150\,Q^5-128\,Q^4+128\,Q^3+150\,Q^2+13\,Q-7)}{(Q^2+Q+1)^3\,(Q+1)}$;
- $g_{33} = \frac{27\,s^6\,(Q^8-4\,Q^7-38\,Q^6+16\,Q^5+131\,Q^4+16\,Q^3-38\,Q^2-4\,Q+1)}{(Q^2+Q+1)^4}$.

Level-subtracting operator :

$$A = \begin{bmatrix} s^3 & \frac{3\,i\,s^4\,(Q-1)}{Q+1} & -\frac{2\,s^5\,(Q-1)^2}{Q^2+Q+1} \\ \frac{s^2\,(Q-1)\,i}{Q+1} & -\frac{(3\,Q^4-5\,Q^3-8\,Q^2-5\,Q+3)\,s^3}{(Q^2+Q+1)\,(Q+1)^2} & \frac{-2\,i\,(Q^5-3\,Q^4-5\,Q^3+5\,Q^2+3\,Q-1)\,s^4}{(Q+1)\,(Q^2+Q+1)^2} \\ -\frac{s\,(Q-1)^2}{Q^2+Q+1} & \frac{-3\,i\,(Q^5-3\,Q^4-5\,Q^3+5\,Q^2+3\,Q-1)\,s^2}{(Q+1)\,(Q^2+Q+1)^2} & \frac{(2\,Q^6-9\,Q^5-18\,Q^4+77\,Q^3-18\,Q^2-9\,Q+2)\,s^3}{(Q^2+Q+1)^3} \end{bmatrix}$$

With these two operators at hand, we can easily compute the closed invariants by taking the traces of the appropriate combinations of the operators, as in theorem 4. Here we reproduce some of these invariants:

$A_3^{\Delta}(0|-2)$:

$$\frac{Q^2\,(Q^2+4\,Q+1)}{(Q+1)^2\,(Q^2+Q+1)^2}$$

$A_3^{\Delta}(0|-1)$:

$$\frac{1}{6\,s^3}$$

$A_3^{\Delta}(0|0)$:

$$\frac{1}{6\,s^6}$$

$A_3^{\Delta}(0|1)$:

$$\frac{Q^6+18\,Q^5-162\,Q^4+394\,Q^3-162\,Q^2+18\,Q+1}{648\,Q^3\,s^9}$$

$A_3^{\Delta}(0|2)$:

$$-\frac{7\,Q^6-276\,Q^5+1284\,Q^4-2138\,Q^3+1284\,Q^2-276\,Q+7}{648\,Q^3\,s^{12}}$$

$A_3^{\Delta}(1|-1)$:

$$-\frac{s^3\,Q\,(Q^6+12\,Q^5-84\,Q^4-182\,Q^3-84\,Q^2+12\,Q+1)}{(Q^2+Q+1)^3\,(Q+1)^2}$$

$A_3^{\Delta}(1|0)$:

$$3$$

$A_3^{\Delta}(1|1)$:

$$-\frac{Q^2-5\,Q+1}{Q\,s^3}$$

$A_3^{\Delta}(1|2)$:

$$\frac{Q^6+66\,Q^5-624\,Q^4+1276\,Q^3-624\,Q^2+66\,Q+1}{54\,Q^3\,s^6}$$

$A_3^{\Delta}(2|0)$:

$$\frac{s^6(1-34Q^9+Q^{10}+288Q^7-645Q^8-34Q-645Q^2+288Q^3+7197Q^6+12630Q^5+7197Q^4)}{(Q^2+Q+1)^4(Q+1)^2}$$

$A_3^{\Delta}(2|1)$:

$$-\frac{(Q^4-70\,Q^3-105\,Q^2-70\,Q+1)\,s^3}{Q\,(Q^2+Q+1)}$$

$A_3^{\Delta}(2|2)$:

$$\frac{Q^4-54\,Q^3+187\,Q^2-54\,Q+1}{Q^2}$$

4.6.2.3 The Calabi-Yau TQFT: an Integrality Result in Degree 3

We now restrict our attention to the embedded Calabi-Yau TQFT. This amounts to considering invariants of a punctured curve X where the level is minus the Euler characteristic of X. In the case of closed invariants, these are:

$$A_3^{\Delta}(g|2g-2).$$

Notice that we have already computed the closed Calabi-Yau invariants for genus $0,1$ and 2.

The Calabi-Yau genus adding operator

$$G^{CY}=GA^{-2}$$

plays a privileged role in this theory. Its eigenvalues $\lambda_1,\lambda_2,\lambda_3$ are the "universal constants" for the TQFT.

Any invariant in the TQFT is expressed as a sum of the appropriate powers of them. In particular, for the closed invariants, we have:

$$A_3^{\Delta}(g|2g-2) = \lambda_1^{g-1} + \lambda_2^{g-1} + \lambda_3^{g-1}.$$

Unfortunately, these constants do not have a nice expression in terms of the variables we have used so far. It is possible nontheless to describe the closed invariants, because they are symmetric functions of the λ_i's. In particular, we have the following integrality result, that answers a conjecture proposed by Bryan and Pandharipande in the first version of [BP04]:

Theorem 8 *For $g \geq 1$,*

$$3^{g-2} A_3^{\Delta}(g|2g-2) = 3^{g-2}(\lambda_1^{g-1} + \lambda_2^{g-1} + \lambda_3^{g-1}) \in \mathbb{Z}[t],$$

where the invariants are expressed in the variable

$$t = \frac{Q^2+Q+1}{Q} = 2\cos u + 1. \tag{4.32}$$

Remarks:

1. This result implies, up to a factor of Q^{2-2g}, an analogous integrality property for the invariants in the variable Q, since the change of variables (4.32) has integral coefficients.

2. The change of variable (4.32) was motivated by the fact that this quantity arises naturally in the computation of the level $(0,0)$ multiplication structure coefficients; it also seems related to the representation theory of S_3.

Proof: the proof boils down to computing the characteristic polynomial of the operator G^{CY}. This is:

$$F(x) = x^3 - (t^2 - 56t + 240)x^2 - \frac{4(t+3)(-72+t^2-60t)x}{3} + \frac{4(t+1)(-72t^2+t^4-60t^3)}{3}.$$

The coefficients of F, which are the elementary symmetric functions in the λ_i's, are polynomials in t. After multiplication by 3, they are polynomials with integer coefficients. Since the closed invariants are the Newton polynomials in the λ_i's, and these can be obtained as polynomials with integral coefficients in the elementary symmetric functions, we obtain that

$$3^{g-1} A_3^{\Delta}(g|2g-2) \in \mathbb{Z}[t].$$

To obtain the sharper result (with the exponent $g-2$ instead of $g-1$) we resort to a more elementary, but less elegant, proof.

We can explicitly solve for the roots of the characteristic polynomial, and obtain the following expression for the eigenvalues:

$$\begin{aligned}
\lambda_1 &= \frac{1}{3}\left(T + \frac{P}{\Theta^{1/3}} + \Theta^{1/3}\right) \\
\lambda_2 &= \frac{1}{3}\left(T + \omega\frac{P}{\Theta^{1/3}} + \omega^2\Theta^{1/3}\right) \\
\lambda_3 &= \frac{1}{3}\left(T + \omega^2\frac{P}{\Theta^{1/3}} + \omega\Theta^{1/3}\right),
\end{aligned} \tag{4.33}$$

where

- $T = t^2 - 56t + 240$ is the trace of G^{CY}.
- $P = 56736 - 27888t + 3388t^2 - 108t^3 + t^4$.
- $\Theta = 13512960 - 9967104t + 2433312t^2 - 234800t^3 + 10512t^4 - 180t^5 + t^6 + 2\sqrt{(-216 + 108t - 63t^2 + t^3)(6048 + 4032t - 3468t^2 + 250t^3 - 3t^4)^2}$.
- ω is a principal cube root of 1.

It is also true that if we denote $\Theta = Q + \sqrt{R}$ in the obvious way, then

$$P^3 = (Q + \sqrt{R})(Q - \sqrt{R}) = Q^2 - R. \tag{4.34}$$

All that we have denoted with capital nongreek letters are polynomials in t with integral coefficients.

Formula (4.34) implies that

$$\frac{P^{3n}}{\Theta^n} + \Theta^n = (Q - \sqrt{R})^n + (Q + \sqrt{R})^n \in \mathbb{Z}[t]. \tag{4.35}$$

To show our integrality result, it suffices to show that, for any positive k,

$$\mathcal{F} := \left(\frac{P}{\Theta^{1/3}} + \Theta^{1/3}\right)^k + \left(\omega\frac{P}{\Theta^{1/3}} + \omega^2\Theta^{1/3}\right)^k + \left(\omega^2\frac{P}{\Theta^{1/3}} + \omega\Theta^{1/3}\right)^k \in 3\mathbb{Z}[t].$$

But this is really quite simple. It is just a matter of expanding $\mathcal{F}$ using Newton's binomial fomula and grouping things in a clever way:

$$\begin{aligned}
\mathcal{F} &= \sum \binom{k}{i} \left(\frac{P}{\Theta^{\frac{1}{3}}}\right)^i \Theta^{\frac{k-i}{3}}(1 + \omega^{k-2i} + \omega^{2i-k}) = \\
&= \sum \binom{k}{i} P^i \Theta^{\frac{k-2i}{3}}(1 + \omega^{k-2i} + \omega^{2i-k}).
\end{aligned}$$

So now we can observe two distinct cases:

1. if $k-2i \not\equiv 0(\mathrm{mod}3)$, then $(1+\omega^{k-2i}+\omega^{2i-k}) = (1+\omega+\omega^2) = 0$.

2. if $k-2i \equiv 0(\mathrm{mod}3)$,then $(1+\omega^{k-2i}+\omega^{2i-k}) = 3$. Hence it suffices to show that the sum of the contributions of these terms is a polynomial with integral coefficients.

To show this, let us pair together the "symmetric" terms that appear in the sum with coefficient $\binom{k}{i}$:

$$P^i\Theta^{\frac{k-2i}{3}} + P^{k-i}\Theta^{\frac{2i-k}{3}}.$$

Supposing, $WLOG$, that $i < k-i$:

$$P^i\left(\Theta^{\frac{k-2i}{3}} + P^{k-2i}\Theta^{\frac{2i-k}{3}}\right) =$$

$$= P^i\left(\Theta^n + \frac{P^{3n}}{\Theta^n}\right) \in \mathbb{Z}[t] \quad \text{by (4.35).}$$

APPENDIX A

COMBINATORICS AND REPRESENTATION THEORY

In this appendix we develop some of the combinatorial and representation theoretic notations and basic facts that we use in this dissertation. The partition notation is pretty universal, and can be recovered in any basic text in combinatorics, or representation theory. For a more detailed account of the few representation theoretic facts we mention, a good reference is [FH91].

A.1 Partitions of an Integer

A partition η of an integer d is a finite sequence of positive integers adding up to d:

$$\eta = (\eta^1, \eta^2, \dots, \eta^r),$$

with

$$\eta^1 \geq \eta^2 \geq \dots \geq \eta^r$$

and

$$|\eta| = \sum_i \eta^i = d.$$

We use the notation $\eta \vdash d$ to indicate that η is a partition of d.

The number r of nonzero integers is called the *length* of the partition η, and denoted $\ell(\eta)$.

It is also convenient to have a notation that groups all equal parts together. By

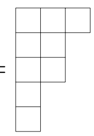
$$((\eta^1)^{m_1} \dots (\eta^k)^{m_k})$$

we denote the partition

$$\eta = (\underbrace{\eta^1, \dots, \eta^1}_{m_1 \ times}, \underbrace{\eta^2, \dots, \eta^2}_{m_2 \ times}, \dots, \underbrace{\eta^k, \dots, \eta^k}_{m_k \ times}).$$

A partition η of the integer d can be canonically represented by a Young diagram. A *Young diagram* is a left justified array of boxes, such that the lenght of the rows does not increase as you go down the diagram. To a partition η we associate the Young diagram whose i-th row is composed of η^i boxes. For example:

$$(3,2,2,1,1) = (3^1 2^2 1^2) =$$



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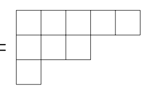
The conjugate partition η' is the partition associated to the reflection along the main diagonal of the Young diagram associated to η. In our example:

$$(3,2,2,1,1)' = (5,3,1) =$$



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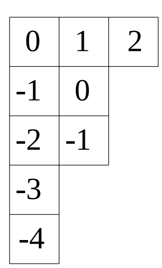
Here are some functions associated to Young diagrams that are useful for our purposes:

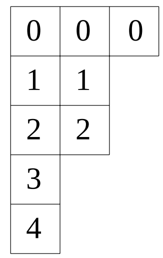
content: for a given box $\square$ in position (i,j) in a Young diagram, we define the content $c(\square) = i - j$. Boxes on the main diagonal will have content 0, boxes above the diagonal will have positive content that measures exacly how many diagonals over the main one they lie on, and so on. The *total content* $c(\eta)$ is defined to be the sum of the content of all boxes in the diagram. If a diagram is symmetric with respect to the main diagonal, then its total content is 0. In some sense, the total content measures the asymmetry of a Young diagram. Figure A.1 illustrates an example.

n**-function:** given a Young diagram, we define the n-function as follows: number all boxes in the first row with 0's, all boxes in the second row with 1's, all boxes in the third row with 2's and so on. Then add all of these numbers to obtain $n(\eta)$. The example in Figure A.2 clarifies the situation.



$c(\eta) = -8$

Figure A.1. The *content* function.



$n(\eta) = 13$

Figure A.2. The n function.

The following obvious formula connects these two quantities:

$$c(\eta) = n(\eta') - n(\eta).$$

hooklength: for a given box $\square$ in a Young diagram, the *hooklength* $h(\square)$ is the length (as in number of boxes) of the hook that has the given box as its north-west corner, as shown in Figure A.3.

The total hooklength $h(\eta)$ is the sum of the hooklengths over all boxes in the diagram (Figure A.4).

A.2 Representation Theory of the Symmetric Group S_d

For any finite group G, it is well known that the number of irreducible representations of G equals the number of conjugacy classes in G. However, in general, there is no canonical correspondence between representations and conjugacy classes.

In the case of the symmetric group S_d conjugacy classes are canonically in bijective correspondence with partitions of the integer d. Partitions of d correspond to Young

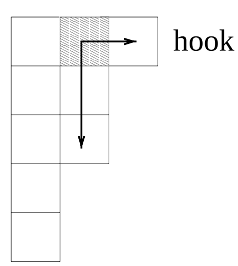


Figure A.3. The hook associated to the shaded box.

7	4	1
5	2	
4	1	
2		
1		

$h(\eta) = 27$

Figure A.4. The total hooklength of a Young diagram.

diagrams, and Young diagrams can be associated to irreducible representations of S_d, thus creating an explicit correspondence.

Let us briefly sketch the construction of Young symmetrizers, that allows us to associate an irreducible representation ρ to a Young diagram η.

First of all number from 1 to d the boxes in the diagram. The order of the numbering is irrelevant, (all possible numberings will yield subspaces of the group algebra related by conjugation) so we may assume the canonical ordering, as represented in Figure A.5. The data of a Young diagram with progressively numbered boxes is called a Young tableaux.

Let S_d act canonically on the set of integers $\{1, \ldots, d\}$ and define two subgroups of the symmetric group S_d associated to the tableaux η:

$$P := \{g \in S_d \text{ that preserve each row in } \eta\},$$

$$Q := \{g \in S_d \text{ that preserve each column in } \eta\}.$$

1	2	3
4	5	
6	7	
8		
9		

Figure A.5. A Young tableaux.

These subgroups correspond to two distinguished elements in the group algebra $\mathbb{C}S_d$, so defined:

$$a_\eta := \sum_{g\in P} \mathrm{e}_g \qquad \text{and} \qquad b_\eta := \sum_{g\in Q} sgn(g)\mathrm{e}_g.$$

Now define the Young symmetrizer:

$$c_\eta = a_\eta b_\eta \in \mathbb{C}S_d.$$

Theorem 9 ([FH91]) *The vector subspace*

$$\mathbb{C}S_d \cdot c_\eta \subseteq \mathbb{C}S_d$$

is an irreducible representation of S_d. Every irreducible representation of S_d can be obtained in this way for a unique partition η.

Examples:

the partition (d): in this case the subgroup P is the symmetric group S_d, Q is only the identity. The symmetrizer is $c_{(d)} = \sum_{g\in S_d} \mathrm{e}_g$ and

$$\mathbb{C}S_d \cdot c_{(d)} = \mathbb{C} \cdot \left(\sum_{g\in S_d} \mathrm{e}_g \right)$$

is the trivial representation.

the partition (1^d): in this case the subgroup Q is the symmetric group S_d, P is only the identity. The symmetrizer is $c_{(d)} = \sum_{g\in S_d} sgn(g)\ \mathrm{e}_g$ and

$$\mathbb{C}S_d \cdot c_{(d)} = \mathbb{C} \cdot \left(\sum_{g\in S_d} sgn(g)\mathrm{e}_g \right)$$

is the alternating representation.

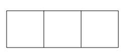
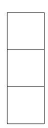
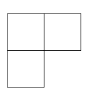
the symmetric group S_3: there are three possible partitions of the integer 3, and three irreducible representations. The complete situation is described in Table A.1.

A.2.1 The Hooklength Formula

Let ρ be an irreducible representation of S_d represented by the Young diagram associated to the partition η. Then the following formula holds:

$$\dim\rho = \frac{d!}{\prod_{\square\in\eta} h(\square)}. \tag{A.1}$$

Table A.1. The irreducible representations of S_3.

Partition	Young Diagram	Symmetrizer	Representation	Dim.
(3)		$id + (12) + (13) + (23) + (123) + (132)$	trivial	1
(1, 1, 1)		$id - (12) - (13) - (23) + (123) + (132)$	alternating	1
(2, 1)		$id + (12) - (13) - (132)$	standard	2

APPENDIX B

COMPARING THE THEORIES

This appendix has the purpose of comparing the theory developed in this thesis with the preceeding theory of Jim Bryan and Rahul Pandharipande in [BP04]. An accurate analysis goes beyond the scope of the section. Rather, we intend to provide a quick reference guide to simplifiy a parallel reading of this thesis and [BP04], by pointing out what we think are the significant relationships and differences of the theories.

B.1 Gromov-Witten Invariants

Gromov-Witten Invariants are intersection numbers on moduli spaces of stable maps. They are interesting mathematical objects by themselves, but they were classically regarded as tools for computations in enumerative geometry. More recently, the interconnection between algebraic geometry and physics has brought renewed interest for particular classes of Gromov-Witten invariants. In [BP01] and [BP03] the local Gromov-Witten invariants of a curve embedded in a Calabi Yau threefold are studied, and used to provide an equivalent and purely mathematical formulation of the physical Gopakumar-Vafa conjecture.

Stable maps and admissible covers are different compactifications of the same moduli space (of maps of degree d from smooth curves to a given curve). The difference in the boundary is given by the fact that stable maps do not put any condition on irreducible components of the source curve that are contracted by the map. Thus there are extremely large-dimensional boundary components (typically larger than the expected dimension of the moduli space) and the moduli space is singular.

It is possible to construct a virtual fundamental class on moduli spaces of stable maps, and effectively do intersection theory on them. However, the enumerative information in the intersection numbers is often clouded by contributions from the “spurious” boundary components. One of the motivating factors in studying intersection numbers on admissible covers is the reasonable hope that they may be more naturally related to enumerative information. In particular, it would be extremely interesting to be able to understand completely the relationship between the two types of intersection numbers, as it would unveil the geometric nature of the boundary contributions for GW invariants.

B.2 A Different Shifting Convention

The partition functions in [BP04] are Laurent series in the formal variable u. All of our generating functions, on the other hand, are simply Taylor series. This difference is due to a different choice of how to organize the invariants in generating function form.

We follow the convention adopted in section 4.2, of accompanying the genus h invariants with the formal parameter raised to the dimension of the genus h moduli space. In [BP04], the following shift is adopted:

$$Z_d(g|k_1,k_2)_{\underline{\eta}} := \sum_{h\in\mathbb{Z}} u^{\star(h)} Z_d^h(g|k_1,k_2)_{\underline{\eta}}, \tag{B.1}$$

where

$$\star(h) = virdim(\overline{M}_h(X_g,d,\underline{\eta})) - d(k_1+k_2).$$

This different conventions allow both theories to have purely trigonometric partition functions. However, we think our convention to be more natural, in the sense that it suffices to set the formal parameter $u=0$ to recover the classical Dijkgraaf TQFT $\mathcal{D}$, as opposed to the statement that the coefficient of the lowest degree term of the partition function coincides with $\mathcal{D}$.

B.3 Relationship between the Partition Functions

Possibly the main evidence for our conjecture for the general degree d Calabi Yau cap (4.5.4) comes from the observation that in low degrees there is a close relationship between the invariants in the two theories. The coefficients for our theory are normalizations of the corresponding $\mathcal{BP}$ coefficients by the totally unramified coefficient:

$$A_d(0|-1,0)_\eta = \frac{1}{s^d d!}\frac{Z_d(0|-1,0)_\eta}{Z_d(0|-1,0)_{(1^d)}}.$$

Our major hope is that this relationship may be in some sense "geometric", possibly expressing exactly the mysterious boundary components contributions to the GW invariant. However, at this stage we only limit ourselves to "observing" the phenomenon.

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