Math 673: worksheet 3

Sheaves

Abstract

We already saw the concept of sheaves as arising from the need of keeping track of local functions. Here we observe that sections of line bundles and vector bundles provide us with another large class of geometrically meaningful examples of sheaves.

1 Sections of line bundles

Consider a line bundle $\pi : L \to X$. For any open set $U \subset X$ we may define the set:

$$\mathcal{L}(U) := \{\sigma : U \to L | \pi \circ \sigma = Id_U\}$$

Since the fibers of L are one dimensional vector spaces (non-canonically isomorphic to $\mathbb{C}$), we may give $\mathcal{L}(U)$ a vector space structure by defining addition and scaling of sections pointwise:

$$(\lambda\sigma_1 + \mu\sigma_2)(x) = \lambda\sigma_1(x) + \mu\sigma_2(x).$$

By defining restriction morphisms in the natural way we view $\mathcal{L} : \mathfrak{Op} \to \mathfrak{Vec}$ as a contravariant functor from the set of open sets of X to the category of vector spaces.

Exercise 1. Spend a little time meditating and convincing yourself that $\mathcal{L}$ gives a sheaf of vector spaces over X.

Exercise 2. If $L = X \times \mathbb{C}$ is the trivial line bundle, show that $\mathcal{L} \cong \mathcal{O}_X$ is isomorphic to the sheaf of functions on X.

Now we highlight some additional structure:

- **$\mathcal{O}_X$ module structure.**

The sheaf of functions $\mathcal{O}_X$ acts on any $\mathcal{L}$: if $f \in \mathcal{O}_X(U)$ and $\sigma \in \mathcal{L}(U)$ we define $f \cdot \sigma \in \mathcal{L}(U)$ by

$$f \cdot \sigma(x) := f(x)\sigma(x).$$

This action is compatible with restriction morphisms and, with this action, we consider $\mathcal{L}$ an $\mathcal{O}_X$-module (or equivalently one says $\mathcal{L}$ is a sheaf of $\mathcal{O}_X$-modules).

- **Locally free.**

Line bundles admit local trivializations, i.e. for every $x \in X$ there is a neighborhood U such that $\pi^{-1}(U) \cong U \times \mathbb{C}$. This isomorphism induces an isomorphim

$$\mathcal{L}(U) \cong \mathcal{O}_X(U),$$

and clearly such isomorphisms are compatible with further restrictions to open subsets of U (one can write $\mathcal{L}_{|U} \cong \mathcal{O}_{X|U}$ to summarize all this). Since in the

category of $\mathcal{O}_X$-modules the free objects are isomorphic to $\mathcal{O}_X^n$, we say that the sheaf of sections of a line bundle is a **locally free sheaf of $\mathcal{O}_X$-modules of rank** 1.
Invertible. We observe that morphisms of line bundles induce morphisms of their sheaves of sections just by composition. Further, one can see that the sheaf of sections of the tensor product of line bundles is the tensor product of the sheaves of sections (in the category of $\mathcal{O}_X$ modules). This is actually a little subtle and I recommend that you meditate a little bit on why this is the case. Therefore sheaves of sections of line bundles are **invertible** in the sense that given $\mathcal{L}$ there exists another sheaf of sections of line bundles (namely the sheaf $\mathcal{L}^\vee$ of sections of $L^\vee$) such that $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$.

Exercise 3. In our discussion we have seen that sheaves of sections of line bundles are locally free sheaves of rank one, and invertible sheaves. In fact these three concepts are equivalent. Spend a little bit of time thinking about how one could prove that from the datum of a locally free sheaf of rank one one can reconstruct a line bundle. Then spend a little more time thinking about how one would prove that invertible sheaves need to be locally free of rank one.

2 Sections of Vector bundles

Consider a vector bundle $\pi : E \to X$. For any open set $U \subset X$ we may define the set:

$$\mathcal{E}(U) := \{\sigma : U \to E | \pi \circ \sigma = Id_U\}.$$

Just as before, we observe that this assignment defines a sheaf of $\mathcal{O}_X$ modules that we denote by $\mathcal{E}$. Vector bundles are also locally trivial, so a trivialization $\pi^{-1}(U) \cong U \times \mathbb{C}^r$ over an open set U allows to identify sections $\sigma \in \mathcal{E}(U)$ with an r-tuple of functions, which means giving an isomorphism

$$\mathcal{E}(U) \cong (\mathcal{O}_X(U))^r .$$

So we see that sheaves of sections of a vector bundle of rank r are **locally free of rank** r.

Just as with line bundles, it takes similar mental yoga to see that any locally free sheaf of $\mathcal{O}_X$ modules of rank r is isomorphic to the sheaf of sections of a vector bundle.

3 Sheaves on $Spec(R)$ associated to an R-module

If $X = Spec(R)$ is an affine variety, then it might not be too surprising that in many cases a sheaf is completely controlled by its global sections: these are called the **quasi-coherent sheaves** and are, in general, the kind of sheaves you will encounter in your daily algebraic geometry life.

Let M be an R module, and we define a sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ as follows. For any $f \in R$, let $U_f = X \smallsetminus V(f)$ be a Zariski open set. The collections of U_f is a basis for the Zariski topology, so if we define what the sections of $\widetilde{M}(U_f)$ are, then everything else follows by standard nonsense. We define $\widetilde{M}(U_f) := M_f$, the localization of M at f. Just as in the case of rings, this consists in formally allowing denominators of f, with the caveat that if any element m of M is annihilated by some power of f, then $m = 0$ in the localized module.

Exercise 4. Understand what is involved in spelling out this definition completely. Make a list of statements that detail what must be shown to prove that $\widetilde{M}$ is indeed a sheaf of $\mathcal{O}_X$ modules on $X = Spec(R)$.

Example 1. Let R be your favorite ring, $X = Spec(R)$ and $M := R \oplus R$.

Then for any open set U, the sections of $\widetilde{M}(U)$ are pairs (f, g), where both f and g are sections of $\mathcal{O}_X(U)$.

Exercise 5. Show that $\widetilde{M}$ from Example 1 is isomorphic to the sheaf of sections of a rank two trivial bundle over $X = Spec(R)$.

Example 2. Let $R = \mathbb{C}[x, y]$ and $M = I_x = \langle x \rangle \subseteq \mathbb{C}[x, y]$. In other words I is the ideal defining the y-axis inside $\mathbb{C}^2$.

If U is an open set that intersects the y-axis, then $x \in \mathcal{O}_X(U)$ remains a non-invertible function on U, hence

$$\widetilde{M}(U) := \langle x \rangle \mathcal{O}_X(U).$$

If U does not intersect the y-axis, then $x \in \mathcal{O}_X(U)$ is an invertible function, and

$$\widetilde{M}(U) := \mathcal{O}_X(U).$$

$\widetilde{M}$ is called an **ideal sheaf**, because for every open set $U \subseteq X$, $\widetilde{M}(U)$ is an ideal in $\mathcal{O}_X(U)$.

Example 3. Let $R = \mathbb{C}[x, y]$ and $M := \mathbb{C}[x, y]/\langle x \rangle$.

If U is an open set that intersects the y-axis, then $x \in \mathcal{O}_X(U)$ remains a non-invertible function on U, hence

$$\widetilde{M}(U) := \mathcal{O}_X(U)/\langle x \rangle \mathcal{O}_X(U).$$

For example if $U_y = \{y \neq 0\}$, then

$$\widetilde{M}(U_y) := \mathbb{C}\left[y, \frac{1}{y}\right].$$

If U does not intersect the y-axis, then $x \in \mathcal{O}_X(U)$ is an invertible function, and

$$\widetilde{M}(U) := 0.$$

Exercise 6. Show that, if I is an ideal in a commutative ring R, the exact sequence of R-modules

$$0 \to I \to R \to R/I \to 0$$

induces an exact sequence of sheaves on $X = Spec(R)$:

$$0 \to \widetilde{I} \to \widetilde{R} \to \widetilde{R/I} \to 0.$$

Give a geometric interpretation of this sequence.

Example 4. Let $R = \mathbb{C}[x]$ and $M := \mathbb{C}[x]/\langle x \rangle$.

Let $U_x = \mathbb{C} \smallsetminus \{0\}$ and let us compute

$$\widetilde{M}(U_x) := M_x = 0,$$

since x, which is now declared to be an invertible element, multiplied to zero any element of M. In general, if U is any open set which does not contain the point $x = 0$, then the sections $\widetilde{M}(U) = 0$.

Let $U_{x-1} = \mathbb{C} \smallsetminus \{1\}$; then we have

$$\widetilde{M}(U_{x-1}) := M_{x-1} = \mathbb{C};$$

since the element $x - 1$ is already invertible in $\mathbb{C}[x]/\langle x \rangle$, nothing happens in the localization. Similarly for any other open set which contains the point 0, the same phenomenon will happen.

$\widetilde{M}$ is called a **skyscraper sheaf** at the point $x = 0$.

4 Coherent Sheaves

In the previous section we defined the notion of a quasi-coherent sheaf on an affine variety. The generalization is handy.

Definition 1. A sheaf $\mathcal{F}$ of $\mathcal{O}_x$-modules on X is **quasi-coherent** if there exists an affine open cover $\mathcal{U}$ of X such that for every $U = Spec(R) \in \mathcal{U}$, $\mathcal{F}_{|U}$ is a quasi-coherent sheaf on U, i.e. $\mathcal{F}_{|U} = \widetilde{M}$, where M is an R-module. Further we say that $\mathcal{F}$ is **coherent** if all such M are finitely generated R-modules.

Exercise 7. Prove that if $E \to X$ is a vector bundle on X, the sheaf $\mathcal{E}$ of sections of E is a coherent sheaf on X.