

Sheaves, Line Bundles and Cartier Divisors

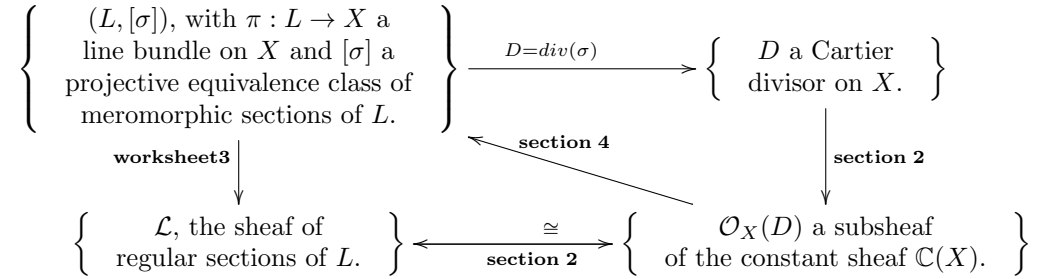
Abstract

We now tie everything together into a nice bow!

1 Roadmaps

To keep everything as simple as possible, let us assume that X is a smooth and projective variety.

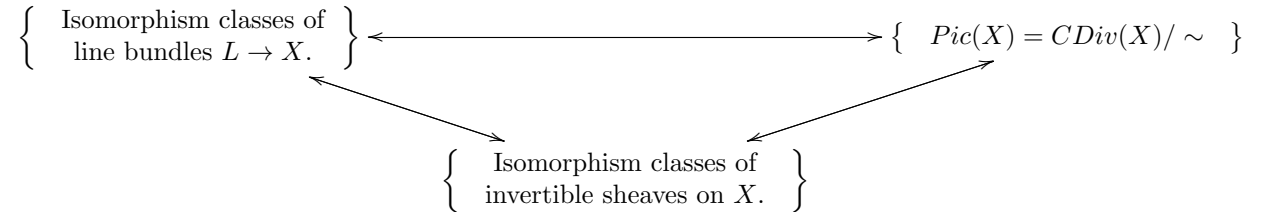
The following two diagrams are the fundamental roadmaps for this worksheet. First we see how we can relate a pair consisting of a line bundle L and an equivalence class¹ $[\sigma]$ of meromorphic sections of L to a Cartier divisor on X . Further, we construct a subsheaf of the constant sheaf $\mathbb{C}(X)$ that we call $\mathcal{O}_X(D)$. This subsheaf is constructed simultaneously with an isomorphism with the sheaf of sections $\mathcal{L}$ of L . Finally we show that $\mathcal{O}_X(D)$ provides patching data to reconstruct the line bundle L .



In Section 3, we observe that:

- two different (projective equivalence classes of) sections σ_1, σ_2 of the same line bundle L give rise to two linearly equivalent divisors $D_1 \equiv D_2$, and $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.
- tensoring line bundles, or their sheaves of sections, corresponds to adding Cartier divisors.

This yields the following diagram, where the arrows are now isomorphisms of groups.



¹ $\sigma_1 \sim \sigma_2 \iff \sigma_2 = \lambda \sigma_1, \lambda \in \mathbb{C}^*.$

2 The sheaf $\mathcal{O}_X(D)$

We are going to try and follow a more pedagogical approach here: instead of defining $\mathcal{O}_X(D)$, and then showing that *et voila*’, it is isomorphic to the sheaf of sections of L , we **discover** $\mathcal{O}_X(D)$ by constructing the most natural trivializations of the sections of $\mathcal{L}$ that we can construct given the datum of a choice of a section σ .

Key observation: while it is not possible to canonically associate a complex number to a point of L , given two points P, Q in L_x , their ratio P/Q is a canonically defined complex number.

Exercise 1. Prove the key observation.

The key observation allows us to construct a map:

$$t : \mathcal{L} \rightarrow \mathbb{C}(X),$$

where $\mathbb{C}(X)$ denotes the field of rational functions on X , which we can view as a constant sheaf on X (this means $\mathbb{C}(X)(U) = \mathbb{C}(X)$ for any non-empty open set U). The map t is defined by

$$t(U) : \mathcal{L}(U) \rightarrow \mathbb{C}(X)(U),$$

$$t(\tau) := \frac{\tau}{\sigma}.$$

It is easy to see that t is an injective map (why?) and therefore $\mathcal{L}$ is isomorphic to its image. We now define

$$t(\mathcal{L}) := \mathcal{O}_X(D). \tag{1}$$

Now let assume

$$D = \sum a_i V_i - \sum b_j W_j,$$

where all $a_i, b_j > 0$ and V_i, W_j are closed, irreducible, codimension one subvarieties of X that admit local equations.

Because the rational functions in $\mathcal{O}_X(D)(U)$ are obtained by dividing by σ and $\text{div}(\sigma) = D$, it follows that such rational functions:

- are allowed poles of order at worst a_i along the subvarieties V_i .
- must have zeros of order at least b_j along the subvarieties W_j .

Exercise 2. Show that another characterization of the sections of $\mathcal{O}_X(D)$ is:

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{O}_X(U) \mid \text{div}(f) + D|_U \geq 0\}.$$

3 Equivalence relations and operations

If we choose two different equivalence classes of sections $[\sigma_1] \neq [\sigma_2]$ of the same line bundle L . Then, since we know that $\sigma_1/\sigma_2 = f$ is a rational function, we have that

$$\operatorname{div}(\sigma_1) - \operatorname{div}(\sigma_2) = \operatorname{div}(f),$$

i.e. the two divisors are linearly equivalent. It follows that the equivalence relation defined on pairs (L, σ) by declaring to pairs equivalent when the sections are sections of the same line bundle corresponds to linear equivalence of Cartier divisors. The sheaf of sections $\mathcal{L}$ clearly doesn't change, but we get different isomorphisms to the $\mathcal{O}_X(D_i)$.

Since an isomorphism of line bundles consists in fiberwise scaling by a non-zero section of $\mathcal{O}_X$ ² we see that the Cartier divisors we obtain as divisors of sections are independent of the choice of a representative in an isomorphism class of line bundles.

If we are given $(L_1, [\sigma_1])$ and $(L_2, [\sigma_2])$, we know that $\sigma_1\sigma_2$ is a section of $L_1 \otimes L_2$. Since we know that

$$\operatorname{div}(\sigma_1\sigma_2) = \operatorname{div}(\sigma_1) + \operatorname{div}(\sigma_2),$$

we have that tensoring of line bundles corresponds to adding Cartier divisors.

4 Reconstructing the line bundle L

First let us assume that $D = V$ is an irreducible, closed, codimension one subvariety of X . Given the sheaf $\mathcal{O}_X(D)$, we construct a line bundle L with a section σ such that $\operatorname{div}(\sigma) = D$.

We start by observing that $\mathcal{O}_X(D)(X \setminus D) = \mathcal{O}_X(X \setminus D)$, and we take this to mean that the line bundle L we want to construct has to be trivial when restricted to the complement of D . I.e. $X \setminus D$ is a chart for a trivialization of L . Now let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a collection of open (affine) subsets such that:

1. $V \subset \bigcup U_\alpha$ (the collection $\mathcal{U}$ covers V).
2. there is $f_\alpha \in \mathcal{O}_X(U_\alpha)$ such that $V|_{U_\alpha} = V(\langle f_\alpha \rangle)$ (The restriction of V to each U_α has a local equation).

Observe that

$$\mathcal{O}_X(D)(U_\alpha) = \frac{1}{f_\alpha} \mathcal{O}_X(U_\alpha).$$

We now define L as follows: a trivialization for L is given by $(X \setminus D) := U_0, \mathcal{U}$.

Transition functions are given by:

$$\Phi_{\alpha,0} := f_\alpha(x).$$

²In particular note that when X is projective the only such sections are constant.

$$\Phi_{\alpha,\beta} := f_\alpha(x)/f_\beta(x).$$

It is immediate to see that these transition functions satisfy the cocycle conditions and therefore define a line bundle L on X . We define a section σ of L by extending the constant section 1 on U_0 using the transition functions above. Such a section vanishes to order one precisely along V which implies that $\text{div}(\sigma) = V = D$.

Now to go to the general case one can proceed in two ways:

1. Realize that advantage of working with the divisor $D = V$ was almost exclusively psychological, and that in fact the same proof will go through for a general divisor, with the only complication that one has to do a bit of bookkeeping in keeping track of the orders of vanishing and zeroes of the section we construct by extending the constant section on the chart $X \setminus \text{Supp}(D)$.
2. Use the fact that addition of line bundles corresponds to tensor product of line bundles, and that taking the dual of a line bundle will produce a minus sign in the corresponding divisors of sections, to piggyback the general case from the basic case we did.

Exercise 3. Spend a little time making sure that these last two points make sense to you.

5 Master Exercises

I think the best way to really understand what we did here is do a couple simple exercises where one can work out all details carefully.

Exercise 4. Let $X = \mathbb{P}^1$, and $D_1 = (1 : 0)$. Construct explicitly the sheaf $\mathcal{O}_X(D)$ and produce an isomorphism with the sheaf of sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. Also show explicitly how to construct the line bundle from the datum of the sheaf $\mathcal{O}_{\mathbb{P}^1}(D)$. Then consider $D_2 = (1 : 1)$. Construct an isomorphism $\mathcal{O}_{\mathbb{P}^1}(D_1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(D_2)$.

Exercise 5. Let $X = \mathbb{P}^2$, and $D = \{Z = 0\}$. Construct explicitly the line bundle determined by $\mathcal{O}_{\mathbb{P}^2}(-D)$. Show that it is isomorphic to the tautological line bundle of $\mathbb{P}^2$.