

MATH 673
Toric Varieties First Semester Review

A **toric variety** is an algebraic variety X together with the algebraic action of an algebraic torus T in such a way that X contains T as a dense open orbit.

Characters and Cocharacters

Character Lattice

$$M := \text{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^n = \{ \{(t_1, \dots, t_n) \mapsto t_1^{k_1} \cdots t_n^{k_n}\} \mid \mathbf{k} \in \mathbb{Z}^n \}.$$

$$M_{\mathbb{R}} := M \otimes \mathbb{R}$$

Cocharacter Lattice or Lattice of One Parameter Subgroups

$$N := \text{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^n = \{ \{t \mapsto (t_1^{l_1}, \dots, t_n^{l_n})\} \mid \mathbf{l} \in \mathbb{Z}^n \}.$$

$$N_{\mathbb{R}} := N \otimes \mathbb{R}$$

Duality

The natural evaluation morphism $M \times N \rightarrow \mathbb{Z}$ is a perfect pairing. Under the above isomorphisms with $\mathbb{Z}^n$ it is given by ordinary dot product. This realizes

$$N = M^{\vee}.$$

Affine Toric Varieties

Torus Embeddings

Given $\mathcal{A} \subset M$ a finite subset of N characters, one gets a map

$$\Phi_{\mathcal{A}} : T \rightarrow \mathbb{C}^N$$

Then

$$X_{\mathcal{A}} := \overline{\Phi_{\mathcal{A}}(T)}.$$

Cones and Semigroups

If $S \subset M$ is a semigroup, then

$$X_S := \text{Spec } \mathbb{C}[S].$$

The toric variety is **normal** (read not too singular) if the semigroup is saturated. Saturated semigroups come from cones. If $\sigma \subset N_{\mathbb{R}}$ is a cone, then

$$X_{\sigma} := \text{Spec } \mathbb{C}[\sigma^{\vee} \cap M]$$

Morphisms of toric varieties

If $f : \sigma_1 \rightarrow \sigma_2$ is an integral map of cones, then $f^\vee : \sigma_2^\vee \rightarrow \sigma_1^\vee$ induces a map on the regular rings of the affine toric variety and hence a map

$$F : X_{\sigma_1} \rightarrow X_{\sigma_2}$$

Of special importance are inclusions of faces $\tau \rightarrow \sigma$, which yield open embeddings of affine toric varieties:

$$X_\tau \subset X_\sigma.$$

Geometric Attributes

An affine toric variety X_σ is **smooth** iff σ is **smooth**, i.e. if the Hilbert basis for the corresponding semigroup is given by the primitive integral vectors of the rays.

If σ is a **simplicial cone**, then X_σ is the quotient of a smooth toric variety by a finite abelian group.

Fans

A **fan** $\Sigma \subset N$ is a collection of cones that are allowed only to intersect along faces. A fan is **complete** if its support is all of $N_{\mathbb{R}}$. Given a fan Σ one naturally constructs a **quasi projective** toric variety X_Σ where:

- a cover by affine charts is given by the toric varieties X_σ , where σ are the maximal cones in Σ .
- patching data is provided by the inclusions of faces $\sigma_1 \leftarrow \tau \rightarrow \sigma_2$.

The toric variety X_Σ is **complete** (think compact in euclidean topology) if and only if the fan Σ is complete. It is **projective** if the fan is the dual fan of a polytope.

The orbit cone correspondence A toric variety X_Σ can be seen as a disjoint union of torus orbits:

$$X_\Sigma = \bigcup_{\sigma \in \Sigma} O_\sigma$$

Further:

- $\sigma_1 \subseteq \sigma_2$ iff O_{σ_2} is in the closure of O_{σ_1} .
- (If the maximal cones of Σ are of top dimension) then the dimension of an orbit is equal to the codimension of the corresponding cone.

Note: if one removes some cones from a fan Σ , the corresponding toric variety is obtained by removing the corresponding torus orbits.

Also: morphisms of toric varieties thus described are given by morphisms of fans (which are integral maps that send cones to cones).

And: just one word on how orbits are obtained. For each cone σ one can obtain a point in the toric variety by looking at homomorphism from the dual cone $\sigma^\vee$. A distinguished point is given by the homomorphism which as “as many 0’s as possible” and then the distinguished orbit is obtained by acting on the distinguished point with the torus T .

Divisors and Sheaves

Weil Divisors A Weil divisor is a formal linear combination of codimension one subvarieties of X . If X_Σ is a toric variety we are mostly concerned with torus invariant divisors: the irreducible torus invariant divisors are indexed by the rays of Σ and denoted D_ρ .

Given a toric variety X_Σ , any character $m \in M$ gives a rational (monomial) function. The order of vanishing (or of poles) of a character along a particular D_ρ is simply given by

$$\nu_m(D_\rho) = \langle m, \rho \rangle$$

The Class group $Cl(X_\Sigma)$ is the quotient of the group of Weil divisors mod the equivalence relation generated by linear equivalence. In other words Weil divisors mod principal divisors. As we saw before we have a short exact sequence:

$$0 \rightarrow M \rightarrow WDiv = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \rightarrow Cl(X_\Sigma) \rightarrow 0.$$

Cartier Divisors Cartier divisors are divisors which are locally principal. For a toric variety X_Σ , a Cartier divisor corresponds to a continuous, piecewise linear integral function φ on each cone of the fan. The Picard group $Pic(X_\Sigma)$ is the quotient of Cartier divisors by Principal divisors.

Recall, that once one has the function φ , the way that one gets the associated divisor is:

$$D_\varphi := \sum_{\rho \in \Sigma(1)} -\varphi(\rho)D_\rho,$$

where I am denoting ρ both a ray and its primitive integral vector. Recall that while it may seem bizarre to put that negative sign in there, there is a good reason, that comes across when we talk about the sheaf of sections of a divisor: by making this choice one can naturally associate a polyhedron to a Cartier divisor, and the lattice points in this polyhedron are the global sections of the sheaf associated to the divisor.

Recall that on a smooth variety Cartier and Weil divisors coincide, and so do the Class and the Picard group. On singular varieties every Cartier

divisor is Weil, but the converse is not true. The standard example is given by just one ruling of a cone.