

Operations on Line Bundles

Abstract

We define the operations of direct sum, tensor product, and dualization of line bundles. In order to talk about a direct sum of line bundles we need to introduce the notion of vector bundle, where we increase the dimension of the fibers arbitrarily. The punchline of this worksheet is that the set of line bundles on a given variety X forms an group under the operation of tensoring.

1 Vector bundles

Now that we are happy with locally trivial families of one-dimensional vector spaces parameterized by a base space X , it is but a short leap to jack up the dimension of the vector spaces. This leads us to the notion of vector bundle.

Definition 1. A **vector bundle** of rank r over X is a space E together with a morphism $\pi : E \rightarrow X$ such that:

1. There is an open cover $\mathcal{U}$ of X such that, for every $U \in \mathcal{U}$,

$$\pi^{-1}(U) \cong U \times \mathbb{C}^r.$$

Denote by $\phi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{C}^r$ such isomorphism.

2. For every $x \in U_1 \cap U_2$, the composition

$$\{x\} \times \mathbb{C}^r \xrightarrow{\phi_{U_1}^{-1}} \pi^{-1}(x) \xrightarrow{\phi_{U_2}} \{x\} \times \mathbb{C}^r$$

is a linear isomorphism, i.e. an element of $GL(r, \mathbb{C})$.

The space X is called the **base** of the vector bundle, and E is called the **total space** of the vector bundle.

Definition 2. A **morphism** of vector bundles is a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & \searrow \pi_1 & \swarrow \pi_2 \\ & X & \end{array}$$

that restricts to a linear function on each fiber. An **isomorphism** is a morphism of line bundles that admits an inverse.

If f is injective we may (by slight abuse of notation) call E_1 a **subbundle** of E_2 . If f is surjective we may (by slight abuse of notation) call E_2 a **quotient bundle** of E_1 .

Note that the kernel and cokernel of a map of vector bundles are in general not vector bundles. In categorical language the category of vector bundles with morphisms as above is not an **abelian category**.

Exercise 1. Consider the trivial line bundle over $L = (\mathbb{C}^2, x, v) \rightarrow X = (\mathbb{C}, x)$ and the morphism $f : L \rightarrow L$ defined by:

$$(x, v) \mapsto (x, xv).$$

Compute kernel and cokernel of f .

Definition 3. A vector bundle E is called **trivial** if $E \cong X \times \mathbb{C}^r$.

Definition 4. A **section** of a vector bundle is a morphism $s : X \rightarrow E$ such that

$$\pi \circ s = Id_X$$

A section s vanishes at the point x if $s(x) = 0 \in E_x$. If a section s is defined everywhere it is called a **global** section. If it is only defined on some open set $U \subset X$ then we call it a **local** section.

Exercise 2. Prove that E is trivial if and only if admits r linearly independent global sections, i.e. $s_1, \dots, s_r$ such that for every $x \in X$, $s_1(x), \dots, s_r(x)$ form a basis for E_x .

Example 1. The embedding of $L \subset \mathbb{P}^1 \times \mathbb{C}^2$ we studied extensively in the first worksheet is an example of a morphism of vector bundles.

2 Direct sum of line bundles

Given two line bundles L_1, L_2 over X , we can form a rank 2 vector bundle denoted $L_1 \oplus L_2$, where each fiber consists of the direct sum of the fiber of L_1 and the fiber of L_2 .

Definition 5. Let $\mathcal{U}$ provide a common trivialization for L_1 and L_2 , and for any double intersection $U_\alpha \cap U_\beta$ denote by $\phi_{\beta,\alpha} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ the cocycle for L_1 and $\psi_{\beta,\alpha} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ the cocycle for L_2 . Then the rank two vector bundle $L_1 \oplus L_2$ is presented using the same trivializing open covers and transition functions

$$\Phi_{\beta,\alpha} : U_\alpha \cap U_\beta \rightarrow GL(2, \mathbb{C})$$

defined by

$$x \mapsto \begin{bmatrix} \phi_{\beta,\alpha}(x) & 0 \\ 0 & \psi_{\beta,\alpha}(x) \end{bmatrix}$$

Exercise 3. Show that sections of $L_1 \oplus L_2$ correspond to pairs of sections (s_1, s_2) , where s_i is a section of L_i .

It requires but a small adaptation to define the direct sum of two vector bundles E_1 and E_2 . Spend a little time to make sure you figure out what such adaptation would be.

3 Tensor product of line bundles

Given two line bundles L_1, L_2 over X , we can produce another line bundle denoted $L_1 \otimes L_2$, whose fiber at each point x should be naturally identified with the tensor product $L_{1,x} \otimes L_{2,x}$.

The key idea here is that the natural isomorphism $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$ is given by multiplication: $z \otimes w \mapsto zw$. This leads us to make the following definition.

Definition 6. Let $\mathcal{U}$ provide a common trivialization for L_1 and L_2 , and for any double intersection $U_\alpha \cap U_\beta$ denote by $\phi_{\beta,\alpha} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ the cocycle for L_1 and $\psi_{\beta,\alpha} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ the cocycle for L_2 . Then the line bundle $L_1 \otimes L_2$ is presented using the same trivializing open covers and transition functions:

$$\Phi_{\beta,\alpha} = \phi_{\beta,\alpha} \psi_{\beta,\alpha} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta).$$

Exercise 4. What line bundle is $\mathcal{O}_{\mathbb{P}}^1(k_1) \otimes \mathcal{O}_{\mathbb{P}}^1(k_2)$ isomorphic to?

Exercise 5. Show that if s_1 is a section of L_1 and s_2 is a section of L_2 , then $s_1 s_2$ is a section of $L_1 \otimes L_2$. However, not all sections of $L_1 \otimes L_2$ need to arise this way. Give an example to illustrate this.

4 The dual of a line bundle

Given a line bundle L , we want to define a line bundle $L^\vee$, whose fiber at each point x should be naturally identified with the dual vector space $L_x^\vee = \text{Hom}_{\mathbb{C}}(L_x, \mathbb{C})$.

Suppose $x \in U_\alpha \cap U_\beta$, and let $v \in L_x$. The point v is identified with two complex numbers z_α and z_β , related by

$$z_\beta = \phi_{\beta\alpha}(x) z_\alpha.$$

Now we want to build $L^\vee \xrightarrow{p} X$ using the same trivialization, so a point φ of $p^{-1}(x)$ is identified with two (one by one matrices valued in) complex numbers w_α and w_β , related by some unknown cocycle:

$$w_\beta = \psi_{\beta\alpha}(x) w_\alpha.$$

But we want $\varphi(v) \in \mathbb{C}$ to be well-defined. For this to happen, we must have:

$$\varphi(v) = w_\beta z_\beta = \phi_{\beta\alpha}(x) z_\alpha \psi_{\beta\alpha}(x) w_\alpha = w_\alpha z_\alpha.$$

Which implies that for every $x \in U_\alpha \cap U_\beta$ we should have $\phi_{\beta\alpha}(x) \psi_{\beta\alpha}(x) = 1$. This leads us to the following definition.

Definition 7. Let $\mathcal{U}$ provide a trivialization for L , and for any double intersection $U_\alpha \cap U_\beta$ denote by $\phi_{\beta,\alpha} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ the cocycle for L . Then the line bundle $L^\vee$ is presented using the same trivializing open covers and transition functions:

$$\psi_{\beta,\alpha} = \frac{1}{\phi_{\beta,\alpha}} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta).$$

It is immediate that $L^{\vee\vee} \cong L$.¹

Exercise 6. What line bundle is $\mathcal{O}_{\mathbb{P}}^1(k)^{\vee}$ isomorphic to?

Exercise 7. The statement *if s is a section of L , then $1/s$ is a section of $L^{\vee}$* is morally right, but not quite right. Why? How do you repair such a statement? If you repair the statement for global sections, how meaningful is such repair? Spend some time just understanding what is going on.

5 The group of line bundles

Given a variety X , the set of isomorphism classes of line bundles $L \rightarrow X$ forms an abelian group, under the operation $\otimes$. In fact:

1. The operation is associative and commutative.
2. The identity element is the trivial line bundle $\mathcal{O}_X \cong X \times \mathbb{C}$.
3. The inverse of a line bundle L is the line bundle $L^{\vee}$.

Exercise 8. Verify the three statements above.

This group is called the Picard group of X , and denoted $Pic(X)$. We have seen this notation when talking about Cartier divisors. Later we will explore the relationship between line bundles and Cartier divisors.

¹We will see that in the more general context of sheaves this equality does not always hold.