Math 673: worksheet 1

Line Bundles

Abstract

Functions from a space X to your favorite field (in our case let us say $\mathbb{C}$) are the essential objects we use to understand the geometry of interesting spaces. Unfortunately many interesting spaces (e.g. all compact complex manifolds, or complex projective varieties) do not have a lot of functions. One way to get around it has been to study local functions, rational functions, and in general a more elaborate algebraic object called the sheaf of functions. It would be nice however to be able to have something global that acts more or less like functions. This role will be played by sections of line bundles. In this first worksheet we give the basic definitions of line bundles and their sections, and look at the most elementary and enlightening class of examples: line bundles on $\mathbb{P}^1$.

Definition 1. A **line bundle** over X is a space L together with a morphism $\pi : L \to X$ such that:

1. There is an open cover $\mathcal{U}$ of X such that, for every $U \in \mathcal{U}$,

$$\pi^{-1}(U) \cong U \times \mathbb{C}.$$

   Denote by $\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}$ such isomorphism.

2. For every $x \in U_1 \cap U_2$, the composition

$$\{x\} \times \mathbb{C} \stackrel{\phi_{U_1}^{-1}}{\to} \pi^{-1}(x) \stackrel{\phi_{U_2}}{\to} \{x\} \times \mathbb{C}$$

   is a linear isomorphism, i.e. a map from $\mathbb{C}$ to $\mathbb{C}$ consisting in multiplication by a non-zero scalar λ_x.

The space X is called the **base** of the line bundle, and L is called the **total space** of the line bundle.

Exercise 1. Draw a picture that illustrates the definition.

Comments and observations:

1. For every $x \in X$, the fiber $L_x := \pi^{-1}(x)$ is isomorphic to $\mathbb{C}$. It is important to notice that such isomorphism is not-canonical. However the 0 element of a vector space is independent of isomorphism, and therefore there is a well-defined subset of L (called the **0-section**) consisting of all points of L that are 0 in the L_x they belong to.

2. By condition 2, given two open sets $U_1, U_2 \in \mathcal{U}$, we obtain a morphism:

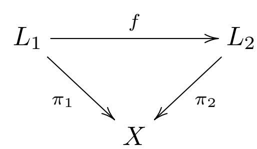
$$\Phi_{2,1} : U_1 \cap U_2 \to \mathbb{C}^*$$

defined by $x \mapsto \lambda_x = [\phi_{U_2} \circ \phi_{U_1}^{-1}(x)]$. So in particular $\Phi_{2,1} \in \mathcal{O}_X^*(U_1 \cap U_2)$. One may show that an open cover $\mathcal{U}$ and a collection $\Phi_{\beta,\alpha} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ for all pairs of intersecting open sets in $\mathcal{U}$ are equivalent information to a line bundle, provided that the **cocycle condition** is satisfied:

$$\Phi_{\alpha,\gamma}\Phi_{\gamma,\beta}\Phi_{\beta,\alpha} = 1$$ [1]

One may think of a line bundle as a family of complex lines parameterized by the base X. Condition 1 tells us that the family is **locally trivial**: while it needs not be globally a product of the space with $\mathbb{C}$, this is true on an open set around any point. Condition 2 tells us that the various trivializations are compatible with each other: they respect fibers and the structure of vector space of each fiber. One seasoned in mathematical lingo might summarize the technical definition by saying that a line bundle is a *locally trivial family of one-dimensional complex vector spaces.*

Definition 2. A **morphism** of line bundles is a commutative diagram



that restricts to a linear function on each fiber. An **isomorphism** is a morphism of line bundles that admits an inverse.

Definition 3. A line bundle L is called **trivial** if $L \cong X \times \mathbb{C}$.

Definition 4. A **section** of a line bundle is a morphism $s : X \to L$ such that

$$\pi \circ s = Id_X$$

A section s vanishes at the point x if $s(x) = 0 \in L_x$. If a section s is defined everywhere it is called a **global** section. If it is only defined on some open set $U \subset X$ then we call it a **local** section.

In other words, giving a section of L consists in picking, for every $x \in X$ a point $s(x) \in L_x$ in the fiber of X. Of course such a choice much be such that the resulting set theoretic function is a morphism in whatever category you want to work over.

Exercise 2. Prove that L is trivial if and only if admits a never vanishing global section.

Example 1. Consider the space $L \subset \mathbb{P}^1 \times \mathbb{C}^2$ defined by

$$L = \{((X : Y), (x, y)) | (x, y) = (tX, tY) \text{ for some } t \in \mathbb{C}\}$$

[1] If one is familiar with Cech cohomology, it is easy to see that a line bundle corresponds to an element of $H^1(X, \mathcal{O}_X^\star)$.

together with the morphism being left projection onto $\mathbb{P}^1$. This is called the **tautological bundle** of $\mathbb{P}^1$ because the fiber over each point of $\mathbb{P}^1$ is precisely the line in $\mathbb{C}^2$ that such point parameterizes.

To show that L is indeed a line bundle we need to exhibit a local trivialization. Let $U_0 = \{X \neq 0\} \subset \mathbb{P}^1$ and $U_\infty = \{Y \neq 0\} \subset \mathbb{P}^1$. Then:

$$\begin{array}{cccc} \phi_0 : & \pi^{-1}(U_0) & \to & U_0 \times \mathbb{C} \\ & ((X:Y),(x,y)) & \mapsto & ((X:Y),x) \end{array}$$

Note that ϕ_0 is invertible since $y = \frac{Yx}{X}$, which is well-defined since $X \neq 0$ on U_0. Similarly:

$$\begin{array}{cccc} \phi_\infty : & \pi^{-1}(U_\infty) & \to & U_\infty \times \mathbb{C} \\ & ((X:Y),(x,y)) & \mapsto & ((X:Y),y) \end{array}$$

Note that ϕ_∞ is invertible since $x = \frac{Xy}{Y}$, which is well-defined since $Y \neq 0$ on U_∞. Finally the transition function

$$\Phi_{\infty,0} = \phi_\infty \circ \phi_0^{-1}$$

maps $x \in \{(X_0 : Y_0)\} \times \mathbb{C}$ to $\frac{Y_0}{X_0}x$, which is a linear isomorphism, since $U_0 \cap U_\infty$ consists of all points of $\mathbb{P}^1$ where both homogeneous coordinates are non-zero.

The **zero section** of L is:

$$Z = \{(X:Y),(0,0))\} \subset L.$$

While this global realization of L as a subset of $\mathbb{P}^1 \times \mathbb{C}^2$ is psychologically quite reassuring, it is important to be able to think of and work with line bundles even when we cannot give such a global realization. In this case, one could "present" L by giving the following data:

- An open cover $\mathcal{U} = \{U_0, U_\infty\}$ of $\mathbb{P}^1$;
- $U_0 \times (\mathbb{C}, u)$ and $U_\infty \times (\mathbb{C}, v)$ with their left projections give the local trivialization; here u and v denote coordinates on $\mathbb{C}$.
- the transition function $v = \frac{Y}{X}u$, defined for all points of $\mathbb{P}^1$ where both homogenous coordinates are different from 0.

Finally, one should keep in mind that U_0, U_∞ are both isomorphic to $\mathbb{C}$ and as such they have affine coordinates $\tilde{y} = \frac{Y}{X}$ and $\tilde{x} = \frac{X}{Y}$, obviously related by the transition function $\tilde{x} = \frac{1}{\tilde{y}}$. So one can describe the line bundle L by giving transition functions between two copies of $\mathbb{C}^2$, as we did in class during the first semester!

To complete this example, let us investigate what the sections of L are. First off, notice that any straight line ℓ not through the origin in $\mathbb{C}^2$ gives you a local section of L which is defined everywhere except at the point of $\mathbb{P}^1$ corresponding to the line through the origin parallel to ℓ. To see whether there are any global sections of L, we look for local sections defined on each chart of the trivialization that match on the intersection.

A local section $s_0 : U_0 \to U_0 \times (\mathbb{C}, u)$ must be of the form $u = s_0(\tilde{y}) = p(\tilde{y})$, for p a polynomial of one variable. Similarly a local section $s_\infty : U_\infty \to U_\infty \times (\mathbb{C}, u)$ must be of the form $v = s_\infty(\tilde{x}) = q(\tilde{x})$, for q a polynomial of one variable.

For $s_{0|U_0 \cap U_\infty} \cong s_{\infty|U_0 \cap U_\infty}$ we must have

$$\tilde{y}u = v = s_\infty(\tilde{x}) = q(\tilde{x}) = q\left(\frac{1}{\tilde{y}}\right),$$

i.e.

$$u = \frac{1}{\tilde{y}} q\left(\frac{1}{\tilde{y}}\right) = p(\tilde{y}).$$

The only possible choices for this to happen are $p = q \cong 0$. Therefore the only global section of L is the zero section.

Exercise 3 (Line bundles on $\mathbb{P}^1$)**.** For any $k \in \mathbb{Z}$, define the line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ to be the line bundle presented by the following data:

- An open cover $\mathcal{U} = \{U_0, U_\infty\}$ of $\mathbb{P}^1$;
- $U_0 \times (\mathbb{C}, u)$ and $U_\infty \times (\mathbb{C}, v)$ with their right projections give the local trivialization; here u and v denote coordinates on $\mathbb{C}$.
- the transition function $v = \left(\frac{X}{Y}\right)^k u$, defined for all points of $\mathbb{P}^1$ where both homogenous coordinates are different from 0.

We will soon see that these are all isomorphism classes of line bundles on $\mathbb{P}^1$. Note that the trivial bundle corresponds to $k = 0$ and that the tautological bundle L that we discussed at length corresponds to $k = -1$.

1. Compute the global sections for every value of k.
2. Observe that this computation shows that different non-negative values of k yield non-isomorphic line bundles.

Example 2. Let us now consider an example of a morphism of line bundles. Consider the global embedding of $L \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ as in Example 1. Recall we are denoting by x, y the affine coordinates on the $\mathbb{C}^2$ factor. We claim that any linear polynomial $p = ax + by$ gives a morphism:

$$\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{P} \mathcal{O}_{\mathbb{P}^1} \cong \mathbb{P}^1 \times \mathbb{C}$$
$$((X : Y), (x, y)) \mapsto ((X : Y), ax + by).$$

This amounts to observing the following:

- The function P is well-defined;
- P preserves the fibers of the line bundles;
- P restricts on each fiber to a linear function.

Exercise 4. Check that P indeed verifies the three bullet points above.