

# Final Project

Math 673

*This project is due on wednesday, May 6th. It will be collected AND graded, for a change.*

## 1 Characteristic classes

### Read on your own Section 13.1.

If you feel a little cheated by the lack of definition of Chern classes, that is comprehensible. Chern classes may be defined in a few different ways none of which actually are useful in terms of working with them. The list of properties that the book gives may then be deduced from the definitions, and those are the “working” properties. Essentially we know that the first Chern class of a line bundle is a divisor of a section of the line bundle, and everything else is typically reconstructed from there. In fact sometimes people take an axiomatic approach (which is what the book does without telling you), i.e. they define Chern classes as families of cohomology classes that satisfy all the properties in page 624. Then one can actually prove that such classes exist and are uniquely determined by that list of properties.

Chern classes and the Chern character are ubiquitous and extremely important in algebraic geometry. The Todd class, in my mind, is motivated by being the “right thing” to make the GRR theorem work. So don’t focus so much on its definition, but look ahead at the role it plays in the theorem. Essentially it is a correction term that is needed for a natural diagram (13.1.8) to actually commute.

**Do exercises 13.1.5, 13.1.6**

## 2 The Atiyah Bott localization theorem

Let  $X_\Sigma$  be a toric variety, or more in general, a variety with a torus action. The Atiyah Bott localization theorem states an isomorphism between the equivariant cohomology of  $X$  and the direct sum of the equivariant cohomology of its fixed loci. Now, if we recall from having discussed the case of  $\mathbb{P}^1$ , this was not the case. The restriction morphism was injective but not surjective. But it was not far from being such... in fact if we were allowed to divide by the equivariant parameters, then we would not have such an

issue. And that's exactly what one must do: allow yourself to invert the equivariant parameters.

**Theorem 1.** *Let  $X_\Sigma$  be a toric variety with torus  $T$  and  $H_T^*(pt.) = \mathbb{C}[t_1, \dots, t_r]$ . Denote by  $F$  the collection of the torus fixed loci, and  $F_i$  each irreducible component of  $F$ . Restriction induces a graded ring isomorphism:*

$$i^* : H_T^*(X_\Sigma) \otimes \mathbb{C}(t_1, \dots, t_r) \rightarrow \bigoplus_{F_i \in F} \mathbb{C}(t_1, \dots, t_r)$$

The inverse function is defined as on each  $F_i$  as:

$$\alpha(t_1, \dots, t_n) \mapsto \frac{i_{F_i,*}(\alpha(t_1, \dots, t_n))}{e^{eq}(N_{F_i})} = \frac{[F_i]\alpha(t_1, \dots, t_n)}{e^{eq}(N_{F_i})},$$

where  $e^{eq}$  denotes the equivariant euler class, and  $N_{F_i}$  is the normal bundle to the fixed locus  $F_i$  in  $X_\Sigma$ .

A very interesting application is to compute intersection numbers. In algebraic geometry, counting the number of points of intersections of two subvarieties of complementary dimension is equivalent to pushing forward the intersection of the two chow classes to a point. Note also that the map  $X_\Sigma \rightarrow pt$  is clearly torus equivariant. Therefore, here is the AB localization strategy for intersecting two chow classes  $a, b$ :

1. Find equivariant cohomology classes  $\alpha, \beta$  whose non equivariant limits are  $a, b$ .
2. Send them forward via  $i^*$ .
3. Multiply them as polynomials on each fixed locus.
4. Send them back via the inverse isomorphism.
5. Push them forward to the class of a point.
6. Take the non equivariant limit (if needed).

The above strategy is summarized in the following residue formula:

**Theorem 2.** *Denote the pushforward to the class of a point by integration. This is a notational legacy coming from de Rham cohomology... Let  $\gamma$  be an equivariant cohomology class (note that  $\gamma$  would be the product  $\alpha\beta$  in the above discussion). Then*

$$\int_{X_\Sigma} \gamma = \sum_{F_i \in F} \frac{\gamma|_{F_i}}{e^{eq}(N_{F_i})}$$

## Exercises

1. Work out explicitly the AB isomorphism for  $\mathbb{P}^1$ .
2. Recall that the class of a line in  $\mathbb{P}^2$  is  $c_1(\mathcal{O}(1))$  and the class of a conic is  $c_1(\mathcal{O}(2))$ . Compute the classical fact that a line intersects a conic in two points via the Atiyah Bott residue formula. Do the computation with at least two different linearizations of the bundles. Can you find some especially efficient choices of linearizations that makes the computations as efficient as possible?