MATH 673

Toric Varieties Cheatsheet

A **toric variety** is an algebraic variety X together with the algebraic action of an algebraic torus T in such a way that X contains T as a dense open orbit.

Characters and Cocharacters

Character Lattice

$$M := Hom(T, \mathbb{C}^*) \cong \mathbb{Z}^n = \{\{(t_1, \ldots, t_n) \mapsto t_1^{k_1} \cdots t_n^{k_n}\} | \mathbf{k} \in \mathbb{Z}^n\}.$$

$$M_{\mathbb{R}} := M \otimes \mathbb{R}$$

Cocharacter Lattice or Lattice of One Parameter Subgroups

$$N := Hom(\mathbb{C}^*, T) \cong \mathbb{Z}^n = \{\{t \mapsto (t_1^{l_1}, \ldots, t_n^{l_n})\} | \mathbf{l} \in \mathbb{Z}^n\}.$$

$$N_{\mathbb{R}} := N \otimes \mathbb{R}$$

Duality

The natural evaluation morphism $M \times N \to \mathbb{Z}$ is a perfect pairing. Under the above isomorphisms with $\mathbb{Z}^n$ it is given by ordinary dot product. This realizes

$$N = M^\vee.$$

Affine Toric Varieties

Torus Embeddings

Given $\mathcal{A} \subset M$ a finite subset of N characters, one gets a map

$$\Phi_{\mathcal{A}} : T \to \mathbb{C}^N$$

Then

$$X_{\mathcal{A}} := \overline{\Phi_{\mathcal{A}}(T)}.$$

Cones and Semigroups

If $S \subset M$ is a semigroup, then

$$X_S := Spec\ \mathbb{C}[S].$$

The toric variety is **normal** (read not too singular) if the semigroup is saturated. Saturated semigroups come from cones. If $\sigma \subset N_{\mathbb{R}}$ is a cone, then

$$X_\sigma := Spec\ \mathbb{C}[\sigma^\vee \cap M]$$

Morphisms of toric varieties

If $f : \sigma_1 \to \sigma_2$ is an integral map of cones, then $f^\vee : \sigma_2^\vee \to \sigma_1^\vee$ induces a map on the regular rings of the affine toric variety and hence a map

$$F : X_{\sigma_1} \to X_{\sigma_2}$$

Of special importance are inclusions of faces $\tau \to \sigma$, which yield open embeddings of affine toric varieties:

$$X_\tau \subset X_\sigma.$$

Geometric Attributes

An affine toric variety X_σ is **smooth** iff σ is **smooth**, i.e. if the Hilbert basis for the corresponding semigroup is given by the primitive integral vectors of the rays.

If σ is a **simplcial cone**, then X_σ is the quotient of a smooth toric variety by a finite abelian group.

Fans

A **fan** $\Sigma \subset N$ is a collection of cones that are allowed only to intersect along faces. A fan is **complete** if its support is all of $N_\mathbb{R}$. Given a fan Σ one naturally constructs a **quasi projective** toric variety X_Σ where:

- a cover by affine charts is given by the toric varieties X_σ, where σ are the maximal cones in Σ.
- patching data is provided by the inclusions of faces $\sigma_1 \leftarrow \tau \to \sigma_2$.

The toric variety X_Σ is **complete**(think compact in euclidean topology) if and only if the fan Σ is complete. It is **projective** if the fan is the dual fan of a polytope.

The orbit cone correspondence A toric variety X_Σ can be seen as a disjoint union of torus orbits:

$$X_\sigma = \bigcup_{\sigma \in \Sigma} O_\sigma$$

Further:

- $\sigma_1 \subseteq \sigma_2$ iff O_{σ_2} is in the closure of O_{σ_1}.
- (If the maximal cones of Σ are of top dimension) then the dimension of an orbit is equal to the codimension of the corresponding cone.

Note: if one removes some cones from a fan Σ, the corresponding toric variety is obtained by removing the corresponding torus orbits.
Also: morphisms of toric varieties thus described are given by morphisms of fans (which are integral maps that send cones to cones).
And: just one word on how orbits are obtained. For each cone σ one can obtain a point in the toric variety by looking at homomorphism from the dual cone $\sigma^\vee$. A distinguished point is given by the homomorphism which as "as many 0's as possible and then the distinguished orbit is obtained by acting on the distinguished point with the torus T.

Toric Varieties as Quotients

Given a fan Σ, a special role is played by the subfan $\Sigma(1)$ of rays of Σ, as they naturally index torus invariant Weil divisors of the toric variety and hence a set of generators for the class group of X_Σ. In fact we have the short exact sequence:

$$0 \to M \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \to Cl(X_\Sigma) \to 0.$$

Recall that the content of this statement is essentially twofold:

- torus invariant divisors are sufficient to generate the class group. This is because removing the torus invariant divisors one is left with a torus which has trivial class group.
- characters give principal divisors. This is in line with our way of thinking of characters as giving "monomial rational functions for the toric variety.

Applying $Hom(-, \mathbb{C}^*)$ to the above exact sequence one gets:

$$0 \to G \to (\mathbb{C}^*)^{|\Sigma(1)|} \to T_N \to 0,$$

which should be thought as follows. There is a big affine space $\mathbb{C}^r$ of which the center term of the sequence is the torus. The group G, which is a "combination of a torus and a finite group acts on this affine space. After throwing away some appropriate coordinate linear subspaces one obtains a good categorical (and sometimes geometric) quotient, which is the toric variety X_Σ. The quotient group is then identified with the torus of the toric variety T_N.
The irrelevant ideal The ideal $B(\Sigma)$ may be constructed in two equivalent ways:

- For each maximal cone σ, consider the product $x^{\hat{\sigma}}$ of the variables corresponding to rays NOT in σ. Then

$$B(\Sigma) = \langle x^{\hat{\sigma}} | \sigma \in \Sigma \rangle$$

This point of view corresponds to the fact that for the affine patch X_σ one wants to use the group action by G to choose a distinguised set of representatives for the orbits where the coordinates of the complementary variables are all 1, and use the variables correponding to rays in σ as local coordinates. In order to be able to do so, one must ensure that there are no points where ANY of the complementary variables are zero.

- Define a **primitive collection** to be a subset C of rays of Σ such that C is not included in the set rays of any one cone, but if you remove any one element of C then the corresponding subset is included in the set of rays of one (maximal) cone. Then

  $$B(\Sigma) = \bigcap_C \langle x_\rho | \rho \in C \rangle$$

  The idea here is that you are throwing away a union of linear coordinate subspaces in the big affine space. It is interesting to note that by doing so one gets a toric variety whose fan is a strict subfan of the fan of $\mathbb{C}^{|\Sigma(1)|}$. In fact this fan $\tilde{\Sigma}$ is a "lifting of Σ, in the sense that there is an incidence and dimension preserving bijection between the cones of the two fans. Then there is a natural map of fans

  $$\tilde{\Sigma} \to \Sigma$$

  that induces a map of toric varieties:

  $$Q : \mathbb{C}^{|\Sigma(1)|} \smallsetminus V(B(\Sigma)) \to X_\Sigma.$$

One then checks that the morphism Q is constant on G-orbits, which then realizes the toric variety as the quotient

$$X_\Sigma = \left(\mathbb{C}^{|\Sigma(1)|} \smallsetminus V(B(\Sigma)\right) //G.$$

Then one can show that the generally this is an **almost geometric quotient** (meaning that it is a geometric quotient on a Zariski open), and that it is a **geometric quotient** (aka the orbit space, aka all orbits are closed) if and only if the fan Σ is simplicial.

Divisors and Sheaves

Weil Divisors A Weil divisor is a formal linear combination of codimension one subvarieties of X. If X_Σ is a toric variety we are mostly concerned with torus invariant divisors: the irreducible torus invariant divisors are indexed by the rays of Σ and denoted D_ρ.

Given a toric variety X_Σ, any character $m \in M$ gives a rational (monomial) function. The order of vanishing (or of poles) of a character along a particular D_ρ is simply given by

$$\nu_m(D_\rho) = \langle m, \rho \rangle$$

The Class group $Cl(X_\Sigma)$ is the quotient of the group of Weil divisors mod the equivalence relation generated by linear eqivalence. In other words Weil divisors mod principal divisors. As we saw before we have a short exact sequence:

$$0 \to M \to WDiv = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \to Cl(X_\Sigma) \to 0.$$

Cartier Divisors Cartier divisors are divisors which are locally principal. For a toric variety X_Σ, a Cartier divisor corresponds to a continuous, piecewise linear integral function φ on each cone of the fan. The Picard group $Pic(X_\Sigma)$ is the quotient of Cartier divisors by Principal divisors.

Recall, that once one has the function φ, the way that one gets the associated divisor is:

$$D_\varphi := \sum_{\rho \in \Sigma(1)} -\varphi(\rho) D_\rho,$$

where I am denoting ρ both a ray and its primitive integral vector. Recall that while it may seem bizarre to put that negative sign in there, there is a good reason, that comes across when we talk about the sheaf of sections of a divisor: by making this choice one can naturally associate a polyhaedron to a Cartier divisor, and the lattice points in this polyhaedron are the global sections of the sheaf associated to the divisor.

Recall that on a smooth variety Cartier and Weil divisors coincide, and so do the Class and the Picard group. On singular varieties every Cartier divisor is Weil, but the converse is not true. The standard example is given by just one ruling of a cone.