Projects

Renzo's math 472

The list of questions/projects below will lead the discussion for this week. Don't be shy, put forth your ideas

1 Dense Sets

Let X be a topological space. A set A is called **dense** if its closure is X.

Warm-up 1. *Give some example of some dense sets in some familiar topologies. What sets are dense in the discrete topology?*

Consider the **finite complement topology** on a set X, where the open sets are ϕ, X and those sets such that their complement is finite.

Problem 1. *What sets are dense in the finite complement topology? Make sure to discuss both the case where X is a finite or an infinite set.*

Consider a topological space (X, τ). Recall that a **base for a topology** is a subset $\beta \subseteq \tau$ such that any open set in τ can be obtained as the union of sets in β.

Problem 2. *Show that if X has a countable base for its topology, then X contains a countable dense subset.*

2 Separating Stuff

A topological space X is called **Hausdorff** if, for any pair of points $x, y \in X$, there are two open sets O_x, O_y such that $x \in O_x$, $y \in O_y$ and $O_x \cap O_y = \phi$.

Problem 1. *Show that if a space X has a **metrizable** topology (i.e. a topology induced by a metric) , then it is Hausdorff.*

A topological space is called **T1** if for any pair of points $x, y \in X$, there is an open set O_x such that $x \in O_x$, $y \notin O_x$.

Problem 2. *Show that being $T1$ is equivalent to the fact that points are closed sets.*

Problem 3. *Can you think of a topological space that is T1 but not Hausdorff?*

Problem 4 (The Zariski Topology). *Consider the following topology on the plane: a set is defined to be closed if it is the zero set of a system of polynomials in two variables, i.e. $C = \{(x, y)$ such that $f_1(x, y) = f_2(x, y) = \cdots = 0\}$.*

1. *Verify that this is indeed a topology on the plane.*

2. *Check that all open sets that are not ϕ are dense.*

3. *Define in the appropriate way the Zariski topology on the line. What topology is it? (i.e. have you seen it before?)*

4. *Is the plane with the Zariski topology Hausdorff? Is it T1?*

3   Product spaces

Let (X, τ) and (Y, σ) be two topological spaces. We define a new topological space $X \times Y$, called the **product (space) of X and Y**. We define a natural topology on $X \times Y$, starting from τ and σ, called the **product topology**.

$X \times Y$ **as a set:** the points of $X \times Y$ correspond to ordered pairs (x, y), where $x \in X$ and $y \in Y$,

A basis for the product topology: a **basic open set** $B \in \beta$ is a set of the form $U \times V$, with $U \in \tau$ and $V \in \sigma$ (i.e. the set of ordered pairs (u, v), with $u \in U$ and $v \in V$). β is a base for a topology called the **product topology**.

Warm-up 2. *Familiarize yourself with the product topology. Draw some pictures, make some examples, understand what the basic open sets look like, and find some examples of open sets in the product topology that are not basic opens.*

Recall that you have two natural maps from a product space called the **projections**:

$$\pi_X : \quad X \times Y \longrightarrow X$$
$$(x, y) \mapsto x$$

$$\pi_Y : \quad X \times Y \longrightarrow Y$$
$$(x, y) \mapsto y$$

Problem 1. *Prove that the product topology is the coarsest topology that makes both projection maps continuous.*

Problem 2. *Show that the torus and the cylinder are product spaces. Show that the product topology coincides with the induced topology from the euclidean topology in $\mathbb{R}^3$.*