The Fundamental Group

Renzo's math 472

This worksheet is designed to accompany our lectures on the fundamental group, collecting relevant definitions and main ideas.

1 Homotopy

Intuition: Homotopy formalizes the notion of "wiggling". Homotopy is a way to compare two functions f,g from the same space X to the same space Y. You imagine a "movie" that takes you continuosly from f (at time 0) to g (at time 1).

homotopy of functions: Let $f, g : X \to Y$ be continuous functions. Then $f \sim g$ (read f is homotopic to g) if there is a continuous function

$$H : X \times I \to Y$$

such that

$$H(x, 0) = f(x), \qquad H(x, 1) = g(x).$$

H is called a homotopy between f and g.

homotopy relative to a subspace: Let $A \subset X$ and $f, g : X \to Y$ be continuous functions such that for any $a \in A, f(a) = g(a)$. Then $f \sim_A g$ (read f is homotopic to g relative to A) if there is a continuous function

$$H : X \times I \to Y$$

such that

$$H(x, 0) = f(x), \qquad H(x, 1) = g(x)$$

and

$$H(a, t) = f(a) = g(a)$$

for all $a \in A, t \in I$.

homotopy equivalence of spaces: Two topological spaces X and Y are homotopy equivalent if there exist functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim Id_X$ and $f \circ g \sim Id_Y$.

contractible spaces: Any space which is homotopy equivalent to a point is called **contractible**.

1.1 Homotopy of based loops

Intuition: We use the notion of homotopy relative to a subspace to define the concept of homotopy of loops. Remembering that for us a loop is a continuous function $\gamma : I \to X$, we define two loops to be homotopic if they are homotopic functions relative to the subspace $0, 1$. Let us repeat this more formally.

Let γ_1, γ_2 be two continuous functions from the closed unit interval to some topological space X such that $\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1) = x$. Then we say that $\gamma_1 \sim \gamma_2$ (read: γ_1 is homotopic to γ_2 as based loops) if there is a continuous function $H : I \times I \to X$ such that:

$$H(s, 0) = \gamma_1(s), \qquad H(s, 1) = \gamma_2(s)$$

and

$$H(0, t) = H(1, t) = x.$$

1.2 Strong deformation retractions

Intuition: The most common appearance of homotopic equivalent spaces will be through deformation retractions. Intuitivelty Y is a strong deformation retraction of X if Y can be "put" inside X and then X can be "continuously sucked into" Y... now let us say just this in mathematese...

A topological space Y is a strong deformation retraction of X if there are two continuous functions

$$i : Y \to X,$$

$$r : X \to Y$$

such that

$$r \circ i = \mathbb{1}_Y$$

$$i \circ r \sim \mathbb{1}_X$$

Note in particular that if Y is a strong deformation retraction of X then X and Y are homotopy equivalent spaces!

2 Category theory

Intuition: OK, this is where fancy mathematics begins...everytime you took a math class, you have been studying a set of mathematical objects (e.g. groups, vector spaces, topological spaces) and their relevant functions (group homomorphisms, linear maps, continuous functions). The way to relate two mathematical theories is to relate the objects (e.g. to any topological space assign a group) and the functions (e.g. to any continuous function assign a group homomorphism) in such a way that composition of functions is respected in such an assignment. This a conceptually rich and systematic way to get a sophisticated topological invariant. Here are a couple more formal definitions, if you care for them :)

A **category** $\mathcal{C}$ is the datum of:

1. a set of objects, usually denoted $Ob(\mathcal{C})$.
2. for each pair of objects $X, Y \in Ob(\mathcal{C})$, a set of morphisms, usually denoted $Hom(X, Y)$.
3. the notion of composition of morphisms, i.e. for every triple of objects X, Y, Z a function $Hom(X, Y) \times Hom(Y, Z) \to Hom(X, Z)$.

Further we require composition of morphisms to be associative and that for each object X we have a distinguished morphism $Id_X \in Hom(X, X)$ which is neutral with respect to composition.

A (covariant) **functor** is a morphism of categories. In other words, given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is the datum of:

1. a function $\mathcal{F} : Ob(\mathcal{C}_1) \to Ob(\mathcal{C}_2)$.
2. for any pair of objects X, Y a function $\mathcal{F} : Hom(X, Y) \to Hom(\mathcal{F}(X), \mathcal{F}(Y))$.

with the natural requirements that identity and compositions are respected, i.e.:

1. $$\mathcal{F}(Id_X) = Id_{\mathcal{F}(X)}$$
2. $$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

3 The fundamental group

The fundamental group is a functor

$$\Pi_1 : \mathcal{PT} \to \mathcal{G}$$

from the category of pointed topological spaces to the category of groups.

Note that the category of pointed topological spaces is just a slight variant of our familiar category of topological spaces. Objects are pairs (X, x_0), where X is a topological space, and $x_0 \in X$ is a point of X. Morphisms between (X, x_0) and (Y, y_0) are continuous functions $f : X \to Y$ such that $f(x_0) = y_0$.

3.1 The function on objects

Given a pointed topological space (X, x_0), the fundamental group $\Pi_1(X, x_0)$ is constructed as follows.

elements the elements are equivalence classes of loops based at x_0. A loop based at x_0 is a continuous function $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x_0$. Two loops α and β are considered equivalent if they are homotopic relative to their endpoints.

operation the operation of composition is just concatenation of loops. In other words:

$$\alpha \star \beta(t) := \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

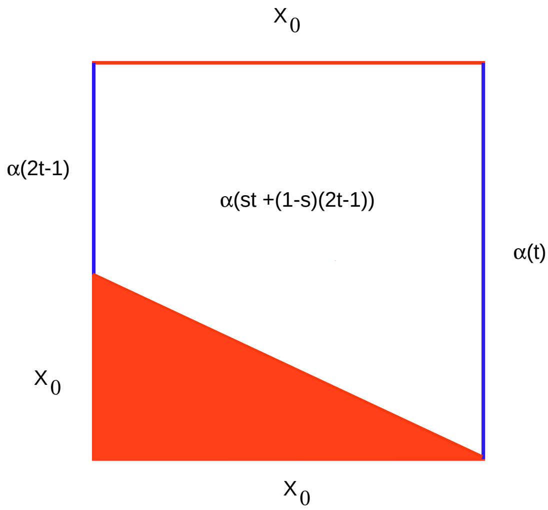
Problem 1. *Check that we have indeed defined a group. You must prove the operation is well defined, associative. That there is an identity element, and that each element has an inverse.*

All of these statements can be proven by pictures. Learn to describe a homotopy of loops in terms of a diagram on a square. For example, suppose that we want to prove that the constant loop ε_{x_0} is the identity element in the fundamental group. We then need to show

$$\alpha \sim \varepsilon_{x_0} \star \alpha \sim \alpha \star \varepsilon_{x_0}$$

A homotopy between the two above loops is a function $H : I \times I \to X$, that can be represented by the following diagram:

We read the diagram as follows. The vertical left hand side of the square is the loop $\varepsilon_{x_0} \star \alpha$. The vertical right hand side of the square is α. The square is a homotopy between the two loops. Everywhere you see red is mapped to x_0. The top white trapeze realizes a continuous morphing of "walking around α twice as fast" into "walking around α at regular speed". In this case I have even written down the function in term of s and t. This is a little confusing, since you should think of both t and s as time. The time t is the time along which you walk along the loop. The time s is the time along which you deform your way of walking around the loop.



3.2 The function on morphisms

Now, given a continuous function $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$, we can define a corresponding group homomorphism:

$$\Pi_1(f) : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$$

$$[\alpha] \mapsto [f \circ \alpha]$$

Problem 2. *Check that indeed $\Pi_1(f)$ is a group homomorphism.*

Problem 3. *Check that Π_1 is indeed a functor between the two above categories.*

4 Some observations about the Fundamental Group

In this section we find some interesting properties of the fundamental group. We already know that the fact that Π_1 is a functor guarantees that it is a topological invariant. In fact, we show that is a homotopy invariant. I.e. in two spaces are homotopy equivalent, them they have isomorphic fundamental group.

Problem 4. *Let $f, g : (X, x_0) \to (Y, y_0)$ be two functions that are homotopic relative to x_0. Then show that*

$$\Pi_1(f) = \Pi_1(g)$$

As a consequence, show that if (X, x_0) and (Y, y_0) are homotopy equivalent relative to the basepoints (figure out what this means!), then

$$\Pi_1(X, x_0) \cong \Pi_1(Y, y_0)$$

What is the fundamental group of a contractible space?

Finally we observe that if a space is path connected, then the choice of the special point is somewhat irrelevant.

Problem 5. *Show that if X is path connected, then for any two points x_0 and $x_1 \in X$ we have*

$$\Pi_1(X, x_0) \cong \Pi_1(X, x_1)$$

This is why one often forgets about the base points and just talks about "the fundamental group of X". However note the subtlety that there is not a unique (or a canonical) isomorphism. You at some point are making a choice. Can you see when?

5 Computing the Fundamental Group

Intuition: OK, so we have decided we really like the fundamental group, because it is a cool topological invariant...but can we actually compute any fundamental group, other the fundamental group of a point, which is more or less by definition the trivial group? We have two tools to do so: one is to use the fact that the fundamental group of two homotopy equivalent spaces is the same; the other is a theorem that tells us how to construct a presentation for the fundamental group of a space if you "chop it down" to simpler pieces of which you do know everything about.

5.1 Homotopy Equivalence

If X and Y are (path connected) homotopy equivalent spaces, then

$$\Pi_1(X, x_0) \cong \Pi_1(Y, y_0)$$

As a consequence, for every n:

$$\Pi_1(\mathbb{R}^n) \cong \Pi_1(D^n) \cong \Pi_1(pt.) = \{e\}$$

Or, for example

$$\Pi_1(cylinder) \cong \Pi_1(annulus) \cong \Pi_1(\mathbb{R}^2 \setminus pt) \cong \Pi_1(S^1)$$

Or, because a segment is contractible, the fundamental group of a space X with some "hair" attached is just the same as the fundamental group of X itself.

Problem 6. *Make sense of the previous sentence. Draw a picture.*

5.2 Seifert-Van Kampen theorem

Theorem 1. *Let X be a path-connected topological space. Let X_1 and X_2 be two open sets such that:*

- $X_1 \cup X_2 = X$;
- $X_1 \cap X_2$ *is non-empty and path connected;*
- $\Pi_1(X_1) =< \alpha_1, \dots, \alpha_n | r_1, \dots, r_p >$;
- $\Pi_1(X_2) =< \beta_1, \dots, \beta_m | s_1, \dots, s_q >$;
- $\Pi_1(X_1 \cap X_2) =< \gamma_1, \dots, \gamma_l | \textit{we don't care} >$.

Then

$$\Pi_1(X) =< \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m | r_1, \dots, r_p, s_1, \dots, s_q, \omega_1, \dots, \omega_l >,$$

where the relations w_i are obtained from the generators of $\Pi_1(X_1 \cap X_2)$ as follows: for each γ, you can write the corresponding loop as a word in α's (say $\gamma = w_\alpha$) and a word in β's (say $\gamma = w_\beta$). Then

$$\omega = w_\alpha (w_\beta)^{-1}$$

Using this theorem, and the fact that the fundamental group of the circle is $\mathbb{Z}$, you can easily deduce:

1. $$\Pi_1(S^2) \cong \{e\};$$

2. $$\Pi_1(8) \cong F_2: \text{ the free group on 2 generators;}$$

3. $$\Pi_1(T) \cong < a, b | ab\bar{a}\bar{b} > \cong \mathbb{Z} \times \mathbb{Z};$$

4. $$\Pi_1(\mathbb{P}^2) \cong < a, b | a^2 > \cong \mathbb{Z}/2\mathbb{Z};$$

Problem 7. *Compute a presentation for the fundamental group of:*

1. *The Klein bottle;*
2. *an arbitrary connected sum of tori;*
3. *an arbitrary connected sum of projective planes.*

6 The fundamental group of the circle

Intuition: We sketch a proof of the computation of the fundamental group of the circle. This is on the one hand a very intuitive and easy to believe result. On the other hand it is reasonably sophisticated to prove. Here goes the statement.

Theorem 2.

$$\Pi_1(S^1) \cong \mathbb{Z}$$

6.1 The Infinite Rotini

The key player in the proof of this statement is the *infinite rotini*, i.e. the continuous function:

$$R : \mathbb{R} \to S^1$$

defined by

$$R(y) = e^{2\pi i y}.$$

Note: the following are elementary yet crucial properties of R.

1. R is surjective.
2. For any $x \in S^1$, the preimage $R^{-1}(x) = \{\ldots, y_n, \ldots\}$ is a countable set, naturally (but not canonically!) in bijection with $\mathbb{Z}$.
3. For every preimage y_n there is an open set $U_{y_n} = (y_n - 1/2, y_n + 1/2)$ such that
$$R_{|U_n} : U_n \to R(U_n)$$
is a homeomorphism.

Sidenote: R is an example of a *covering map*. Every time we have a function $f : X \to Y$ with properties 1. and 3. we say that f is a covering map or that Y is a *covering space* of X. This is the beginning of a beautiful story... that will be left for another time.

6.2 Lifting

Given a function $f : X \to S^1$ a *lifting* of f is a function $\tilde{f} : X \to \mathbb{R}$ such that:

$$f = R \circ \tilde{f}.$$

What is crucial to us is that if X is either an interval or a square, then:

1. A lifting always exists.

2. Once you specify the image of just one point for $\tilde{f}$, then the lifting is unique.

The proof of this statement is an excellent exercise for you to think about. You have to exploit property 3. to lift locally (locally R is a homeo and hence it has an inverse function!). Then you use compactness of the interval and of the square to show that you can lift globally.

6.3 A silly but useful way to think of $\mathbb{Z}$

Denote by Γ the group defined as follows:

set homotopy equivalence classes (relative to the enpoints) of paths starting at 0 and ending at some integer.

operation composing two paths γ_1 and γ_2 is defined as follows: go along γ_1 twice as fast. Say that the endpoint of γ_1 is n. Then translate γ_2 by n and go along that path twice as fast. Here is in math notation:

$$\gamma_1 \star \gamma_2(t) := \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) + \gamma_1(1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Since $\mathbb{R}$ is contractible, it is easy to see that there is only one equivalence class of paths for any given integer endpoint. Upon ε more thought it should be reasonably evident that the operation we defined corresponds to additions of integers. Therefore our mystery group Γ is just a funny way of talking about $(\mathbb{Z}, +)$.

6.4 A natural isomorphism

Now we have natural ways to go from Γ to $\Pi_1(S^1)$ and back:

Projection Given a path $\gamma \in \Gamma$, $R \circ \gamma$ is a loop in S^1.

Lifting Given a loop $\alpha \in \Pi_1(S^1)$, the (unique) lifting $\tilde{(}\alpha)$ of α starting at 0 is a path in Γ.

The major issue here is to show that the map *Lifting* is well defined (or if you want, that *Projection* is injective). But this is a consequence of the fact that continuos maps from squares lift as well! Homotopies of paths are in particular maps from a square, and therefore if two loops downstairs are homotopic, so are their lifts, and this in particular implies that their endpoints agree.

The final thing to check is that *Lifting* and *Projection* are group homomorphisms and that they are inverses of each others. And that's another good exercise for you!

6.5 A generalization

There is a nice generalization of this theorem that goes as follows.

Theorem 3. *Let X be a contractible space and G be a finite group acting on X in such a way that the quotient map*

$$X \to X/G$$

is a covering map. Then

$$\Pi_1(X) \cong G$$

Note that in particular this applies to some of our old friends:

1. $\mathbb{P}^2 = S^2/\mathbb{Z}_2$.
2. $T = \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$