LECTURE PLAN FOR MATH 472
INTRO TO TOPOLOGY
Fall 2010

1 Aug 23

Announcements

1. Exams
2. Hwk
3. Office times/talk to each other first
4. Feedback

Draw pictures about things that are equal or different in different geometries. Explore:

- euclidean 3d (LENGTH, ANGLES, AREA, NUMBER OF SIDES)
- euclidean 2d (CLOCKWISE-COUNTERCLOCKWISE ORDERING)
- topography (ANGLES, PROPORTIONS)
- topology (NUMBER OF PIECES, NUMBER OF HOLES, EULER, DIMENSION)
- homotopy theory

History: Felix Klein *Erlangen Programme, 1872*

1. Continuity. Generalizing our old notion. The concept of equality in topology.
2. First invariants: connectedness, compactness.
3. Ways of constructing new spaces from old ones. Products, Quotients.
4. Study of compact surfaces. Some more sophisticated invariants: Euler char., orientability, genus.
5. Some algebraic topology. The fundamental group. The concept of homotopy.
6. Advanced topics if time permits.

2 Aug 25: Life with a metric

Define a distance function.

Example 1. *1. euclidean.*

2. taxicab.

3. *discrete.*
4. *max* $|f - g|$.
5. *min* $|f - g|$ *(NOT)*
6. *max* $|x_i - y_i|$
7. *max* $|x_1 - y_1|$*(NOT)*
8. $(x - y)$ *(NOT- symmetric, positive)*
9. *do examples with two or three points.*

Define continuity. Do some examples. Define open balls, limit points.

Exercise 1. *Draw open balls for all the above metrics.*

Exercise 2. *Euclidean metric. Which are limit points:*

- *fuzzy blob and a point on the boundary.*
- *point in the interior of a blob.*
- A *= isolated point outside a blob.*

Exercise 3. *What are limit points for sets in the discrete metric?*

Open and closed sets. Examples in the euclidean metric.

Exercise 4. *Prove that a set is open/closed in the euclidean metric if it is open/closed in taxi-cab or max*

Exercise 5. *What are open and closed sets in the discrete metric?*

Properties:

- $\phi; X$ are open/closed.
- arbitrary union of opens is open.
- finite intersection of open is open. (Q: how about arbitrary intersection?)
- complement of an open is closed and viceversa

Theorem 1. *f is continuous if and only if $f^{-1}(open)$ is open. Same with closed.*

Exercise 6. *Any relation between continuity and the image of open sets?*

3 Aug 27

Define topological space and a topology (X, τ). Can define by opens or by closed.

Example 2.

- *stupid (trivial) topology. Discrete topology.*
- *topology induced by a metric.*
- *topologies on a set with two points.*
- *the topology that loves one point.*
- $[a, b)$ *on* $\mathbb{R}$*(NOT).*
- *finite complement topology.*
- *induced topology on a subset.*

Exercise 7. *Which of these are topologies on* $\mathbb{R}$*?*

- $A = \{(-\infty, a)\}$
- $B = \{(b, \infty)\}$
- $A \cup B$

3.1 Basic concepts

1. neighborhood;
2. interior point/interior of a set;
3. limit points/closure of a set.

Exercise 8. *Give two equivalent definitions for interior and closure.*

4 Aug 30

Discuss the induced topology on a subset.

Exercise 9. *Let* X *have the stupid topology, and* $x \in X$*. For what subsets of* X *is* x*:*

- *an interior point?*
- *a limit point?*

What is the smallest neighborhood of X*? And of the complement of* X*?*

Define the concept of fineness/coarseness of a topology.

Exercise 10. *What are the finest and coarsest topologies you can define on a set X?*

Exercise 11. *Partial order the following topologies on $\mathbb{R}$:*

- *stupid;*
- *discrete;*
- *finite complement;*
- *euclidean;*
- $(-\infty, a)$

Basis for a topology in two ways:

1. Collection of sets such that every open set is a union of basic sets.
2. Every point in an open set U has a basic open neighborhood in U.

Exercise 12. *Show that open balls are a basis for euclidean. Ask them for a basis for discrete topology.*

Define continuity.

Example 3. *Any function to a space with the stupid topology is continuous.*
Any function from a space with the discrete topology is continuous.
$\sin(x)$ *on* $\mathbb{R}$ *with the finite complement topology is not continuous.*
But $1/x$ *arbitrarily defined at* 0 *is!*
Take function $y = [x]$*: it is continuous with topology* $(-\infty, a)$*, but not with* (a, ∞)*.*

Exercise 13. *Assign to think about for friday: identity function on $\mathbb{R}$. When continous with all possible pair of assignments of topologies on $\mathbb{R}$ we have seen.*

5 Sep 2nd, 4th

Theorem 2. *The composition of continuous functions is continuous*

Theorem 3. *f is conitinuous iff the preimage of any element of a base for the topology on Y is open in X.*

Define a homeomorphism in two equivalent ways:

1. f continuous with inverse continuous.
2. f a bijiection inducing a bijection on the topologies.

Note that a function being a homeomorphism depends on the topology on the set! Even the identity function on a set is not necessarily a homeomorphism.

Example 4. *Open interval, $\mathbb{R}$. Open disc, plane, sphere minus a point. Sphere and cube.*

Theorem 4. *A homeo between the boundary of a disc to itself can always be extended to the whole disc.*

6 Sep 8

More example of homeomorphisms and not.

Homeos:
Square and Circle.
Sphere and cube.
Circle and interval $[0, 1)$ with appropriate topology. Circle and one-point compactification of the open interval.

Exercise 14. *Prove that $\mathbb{Z}$ and $\mathbb{Q}$ are homeo with the discrete topology, but $\mathbb{Q}$ anr $\mathbb{R}$ are not. Prove that $\mathbb{Z}$ and $\mathbb{Q}$ are NOT homeo when given the induced topology from the euclidean on $\mathbb{R}$. Prove that $\mathbb{R}$ with the finite complement topology is not homeo to $\mathbb{R}$ with the half line topology.*

Exercise 15. *Let us try to prove that $(0, 1)$ and $[0, 1]$ are not homeomorphic to each other.*

Exercise 16. *Subdivide in homeomorphism classes the letters of the alphabet.*

7 Sep 10-13-15

Get some projects going. Worked in class on proj1.pdf. Explored dense sets, separation axioms and the product topology.

8 Sep 17-20

8.1 Connectedness and path connectedness

Idea: want to model the notion of being "only one piece". Two ideas arise:

1. Joining points by paths.
2. Idea motivated by subspace topology.

Anticipate that the two notions will not be equivalent!

8.1.1 Connectedness

Equivalent definitions. X is **disconnected** if

1. There is a proper subset of X which is open and closed.
2. X can be decomposed into the disjoint union of two nonempty open (and hence closed) subsets.
3. You can decompose $X = A \cup B$ with $\bar{A} \cap B = \phi$ and $\bar{B} \cap A = \phi$
4. X admits a continuous non-constant map to a discrete space.

Then you define connected if not disconnected.

Example 5. *1. $\mathbb{R}$ is connected. This is tricky to prove. Assume a disjoint decomposition. Take a point in each set (call them a and b and assume a less than b) and consider the sup of the points of A that are less than b. Look at all possible cases.*

2. $\mathbb{Q}$ is NOT connected.

Theorem 5. *The continuous image of a connected is connected.*

Theorem 6. *Connectedness is a topological invariant.*

Theorem 7. *If Z is dense in X and Z is connected, then X is connected. As a corollary any subset between Z and its closure is connected.*

Define **connected component**: a connected subset of X maximal with respect to being connected. Note that by the theorem above a connected component is always closed. Is it open as well?

Theorem 8. *$X \times Y$ connected iff X and Y are.*

The difficult direction is handled this way:

Theorem 9. *X is the union of a collection of connected spaces, such that any two spaces in the collection intersect, then X is connected.*

8.2 Path Connectedness

Define path connectedness.

Theorem 10. *path connected implies connected.*

Theorem 11. *connected open in $\mathbb{R}^n$ is path connected.*

Note: for all not too pathological spaces the two notions are equivalent.

Exercise 17. *Give an example of a connected space which is not path connected.*

Typical example is the comb with teeth accumulating to one side, and a single dot on that side.

Theorem 12. *$X \times Y$ path conn iff X and Y are.*

9 Sep 22-24

9.1 Compactness

Intuitive idea: compactness measures the fact a set is "solid. In $\mathbb{R}^n$, compact is equivalent to closed and bounded. You can extend this definition to all metric spaces, but of course we have to be able to define things in such a way that they work even when there is no metric!!!

Define compactness: from any open cover, you can extract a finite subcover.

Exercise 18. $\mathbb{R}$ *not compact.* $(0, 1)$ *not compact.*

Exercise 19. *What subsets of X are compact if X has the stupid topology? Discrete? Finite complement topology?*

As usual, the first step is checking that we have defined something that generalizes our notion in metric spaces. hence the

Theorem 13 (Heine-Borel)**.** *Cpt in $\mathbb{R}^n$ iff closed and bounded.*

Proof: Easy direction: If cpt then closed and bounded. Prove by contradiction.

Hard direction: If closed and bounded then cpt. Take arbitrary cover. Subdivide in squares of radius 1. Find one where you can't find finite subcover. Iterate. Get succession of nested closed sets with radius going to 0. Must converge to a point, which is a limit point, and therefore in the set. There is an open set in the collection containing that point. Therefore it contains one of the subdivided boxes (if radius smalll enough). Contradiction.

9.2 Theorems about compactness

Theorem 14. *Image of cpt is cpt.*

Corollary. Compactness is a topological invariant.

Theorem 15. *Closed subset of cpt is cpt.*

Theorem 16. *Compact in Haussdorff is closed.*

Theorem 17 (Bolzano-Weierstrass)**.** *An infinite subset of a compact space has a limit point.*

Theorem 18. *$X \times Y$ is cpt iff X and Y are.*

10 Oct 13

Given a function

$$f : X \longrightarrow Y,$$

where one is a topological space and the other is not, how to induce a topology!

10.1 Y has a topology

Define τ_X to be the coarsest topology that makes f continuous. This corresponds to throwing in all preimages of open sets of Y.

Q: have we seen this already? Yes, with the subspace topology!!

10.2 X has a topology

Define τ_Y to be the finest topology that makes f continuous. I.e. define all sets whose preimage is continuous.

Exercise 20. *Step function. Use euclidean on domain and codomain and see what topologies happen!*

- *In one case get basis* $[n, m)$
- *In the other get basis given by* $\{x \notin \mathbb{Z}\} \cup \mathbb{Z}$

Example 6. *Glue the segment [0,1] to get a circle.*

11 Oct 15

Quotient topology. See LectureplanT, page 8.

12 Nov 12

Define homotopy of maps.

Examples:

- Any two maps $X \to \mathbb{R}^n$ are homotopic.
- Any two maps $pt. \to X$ are homotopic only if X is path connected.
- Question: are any two maps $X \to S^n$
- $f : e^{ix}, g : e^{i(x+\pi)}$ are homotopic.
- embedding of meridian circle on torus and constant map are not homotopic.
- 4 sides of polygon for torus and constant map are homotopic.

Results about homotopies:

1. Homotopy is an equivalence relation.
2. composition of homotopic maps are homotopic.

13 Nov 15

Homotopy equivalence of spaces: $X \sim Y$ if I have a pair of maps whose composition is homotopic to the identity.

Homotopy equivalence is an equivalence relation.

Examples:

1. $\mathbb{R}^n$ and point. Disc and point.
2. circle and cylinder and plane minus a point.
3. 8 and plane minus two points.

Homotopy relative to a subset.

Deformation retraction: i, r such that $ir \sim 1$, $ri = 1$.

Exercises: 12

Homotopy equivalence of spaces: $X \sim Y$ if I have a pair of maps whose composition is homotopic to the identity.

Homotopy equivalence is an equivalence relation.

Examples:

1. $\mathbb{R}^n$ and point. Disc and point.
2. circle and cylinder and plane minus a point.
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Deformation retraction: i, r such that $ir \sim 1$, $ri = 1$.

14 November 17- 19

Worksheet on the fundamental group.