Final Project

The classification of compact surfaces

Rules of the Game

This worksheet is your final exam for this course. You will work on these problems in class. Douglas will be there to help you for the last two weeks of classes, take advantage of him. You can do further work at home, by yourselves, with others...so long as you then write down your own answer. Please take care in writing your answers well, both from the point of view of content and exposition. You will need to write up:

1. *Problems 5, 8, 9, 15, 16, 18, 19, 20, 21, 23;*
2. *A problem of your choice among problems 2, 3 and 4;*
3. *A problem of your choice between problems 12 and 13;*
4. *Most importantly! You must write up a sketch of proof of Theorem 1. This involves putting together all that is done in the worksheet. Note that in this sketch you should not re-prove things that you have proved in various exercises, but just refer to the appropriate exercises.*

!!! EMAIL YOUR ANSWERS TO renzo@math.colostate.edu NO LATER THAN DECEMBER 11th 2012 !!!

The goal of this project is to understand and prove the theorem of classification of compact surfaces. We briefly recall that a **surface** is a Hausdorff topological space S which is locally homeomorphic to $\mathbb{R}^2$. We also recall that **locally homeomorphic** means that for every point $x \in S$, there exists a neighborhood U_x of x which is homeomorphic to an open disc in the plane.

Theorem 1. *Any connected compact surface is homeomorphic to one of the following:*

1. S^2*: a sphere.*
2. $T^{\#g}$*: the connected sum of* g *tori.*
3. ${\mathbb{P}^2}^{\#m}$ *the connected sum of* m *projective planes.*

Further, any two distinct surfaces from the above lists are not homeomorphic.

1 Connected Sums

Let S_1 and S_2 be two compact, connected surfaces. Let D denote an open two dimensional disc. We construct a new surface which we call the **connected sum** of S_1 and S_2 (and denote $S_1 \# S_2$) as follows:

- remove an open disc from both surfaces;
- choose a homeomorphism φ from the boundary of $S_1 \setminus D$ to the boundary of $S_2 \setminus D$;
- consider the surface obtained as a quotient space of $S_1 \setminus D \cup S_2 \setminus D$ under the equivalence relation $x \sim \varphi(x)$.

Informally what you are doing is gluing the two surfaces along the boundaries introduced by removing discs.

Problem 1. *What happens if you choose S_1 to be a sphere?*

Now we need to show that this operation is well defined, i.e. that the homeomorphism class of the resulting surface is independent of the choices you have made. We break this into two exercises:

Problem 2. *Prove that if you choose a different homeomorphism (say ψ) from the boundary of $S_1 \setminus D$ to the boundary of $S_2 \setminus D$, then the surface obtained by gluing along the equivalence relation $x \sim \psi(x)$ (let us denote this surface by $S_1 \#_\psi S_2$) is homeomorphic to the surface obtained by gluing along the equivalence relation $x \sim \varphi(x)$ (denoted $S_1 \#_\varphi S_2$).*

Hint: consider a little annulus around the boundary of S_1 and think of it as the product space $A = S_1 \times [0, 1]$. Construct two homeomorphisms Φ and Ψ from A to itself; Φ such that

$$\Phi(x, 0) = (x, 0) \qquad \Phi(x, 1) = (\varphi(x), 1),$$

while Ψ such that

$$\Psi(x, 0) = (x, 0) \qquad \Psi(x, 1) = (\psi(x), 1).$$

Use this two homeomorphisms to construct a homeomorphism (note: it has to be defined piecewise!!) between $S_1 \#_\varphi S_2$ and $S_1 \#_\psi S_2$.

Problem 3. *Prove that the homeomorphism class of the connected sum of two surfaces does not depend on where you remove the open discs from the surfaces.*

Hint: here it is sufficient to prove that given a surface S and two discs D_1 and D_2 on the surface, then $S \setminus D_1 \cong S \setminus D_2$. In order to do so, consider a simple path connecting the centers of the two discs, and a small closed thickening T of it that contains the two discs. Define (geometrically please! No monstrous attempt at equations) a homeomorphism Φ from $T \setminus D_1$ to $T \setminus D_2$ that is the identity on the boundary of T. Then use Φ to define (in a piecewise way) a homeomorphism proving $S \setminus D_1 \cong S \setminus D_2$.

Finally we wish to show that an operation which is a slight modification of a connected sum can in fact be thought as a connected sum with a torus.

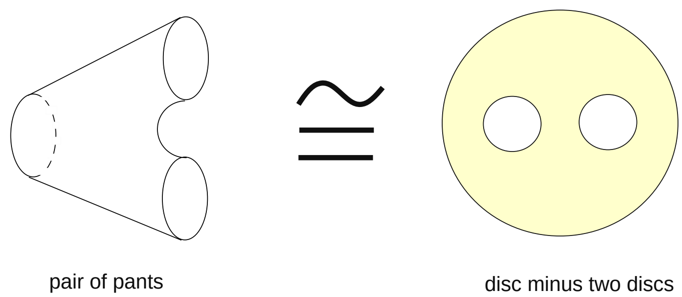

Figure 1: A pair of pants and a disc minus two discs. The surface of the pants should also be shaded but it was too hard to do on the computer, sorry...

Problem 4. *Let S be a connected surface, and consider two open discs D_1 and D_2 on S. Then attach a cylinder to $S \setminus \{D_1, D_2\}$ by identifying the two components of the boundaries of $S \setminus \{D_1, D_2\}$ to the boundaries of the cylinder. Call the resulting surface $\tilde{S}$. Prove that*

$$\tilde{S} \cong S\#T,$$

where T denotes a torus.

Hint: first prove that a pair of pants is homeomorphic to a disc minus two discs (see figure 1). Then use such a homeomoprhism to write down a piecewise defined homeomorphism between $\tilde{S}$ and $S\#T$.

2 Identification Polygons

A way to represent a surface is by giving an identification polygon. So first let us define formally what this is.

An **identification polygon** is:

- a polygon with an even number of sides. Let us say there are $2n$ sides.
- an arrow choosing a direction for each side.
- a labeling of each edge with a letter chosen between $a_1, a_2, \ldots, a_n$ in such a way that each letter appears exactly twice.

We obtain a surface from an identification polygon as follows: informally, we "zip together" the edges with the same label, in the direction prescribed by the arrows. Formally, for every $i = 1, \ldots, n$ let ϕ_i be a homeomorphism between the two edges labeled by the letter a_i, in such a way that this homeomorphism respects the direction of the corresponding arrows. Define an equivalence relation on the polygon by imposing $x \sim \phi_i(x)$ for every i. Then we define S to be the quotient space of the polygon with respect to the above equivalence relation.

Problem 5. *Prove that any surface coming from an identification polygon is compact.*

Problem 6. *A quick and dirty way to give an identification polygon is by just giving a word in the a_i's and in $\bar{a}_i$'s, with $2n$ letters, and in such a way that for each i the letter a_i or $\bar{a}_i$ appears exactly twice. Find a natural function between the set of such words and the set of identification polygons.*

Problem 7. *What is the word associated to the identification polygon of a sphere? Of a torus? What surface do we obtain from the identification polygon corresponding to the word $a_1 a_2 \bar{a}_1 a_2$?*

Problem 8. *Find all distinct identification polygons with two or four sides and identify the corresponding surfaces.*

Identification polygons play well with connected sums. Convince yourself that if S_1 is represented by an n-gon P_1 and S_2 is represented by an m-gon P_2, then the connected sum $S_1 \# S_2$ can be represented by an $n+m$-gon corresponding to just concatenating the words for P_1 and P_2.

Problem 9. *Prove:*

- $\mathbb{P}^2 \# \mathbb{P}^2 \cong$ *Klein Bottle*;
- $T \# \mathbb{P}^2 \cong \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$.

Deduce that any connected sum of tori and projective planes is actually homeomorphic to either a connected sum of only tori, or a connected sum of only projective planes.

3 All surfaces come from Polygons

We assume, without a formal proof, a fairly plausible fact: *any compact connected surface admits a finite triangulation.* Intuitively **triangulation** is just a way to "subdivide the surface into triangles, in such a way that adjacent triangles share entire sides (see Figure 2). A (slightly) more formal way to say that a triangulation is a graph on a surface whose dual graph is trivalent.

Given a triangulated surface, one can construct an identification polygon representing the surface as follows:

1. label all sides of all triangles with distinct letters, and give each side an orientation (arbitrarily);
2. cut along all sides to obtain a bunch of disjoint (labeled) triangles;
3. reglue the triangles appropriately to obtain a polygon.

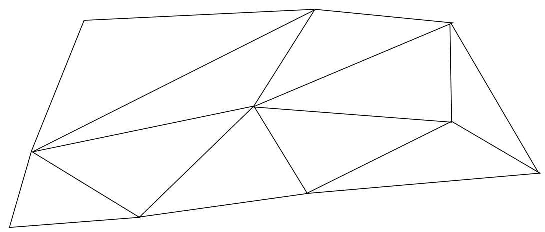

Figure 2: A piece of a triangulation on a surface

The third point needs to be understood a little more concretely. In order to do so.

Problem 10. *Illustrate this strategy for a sphere. Choose two different triangulations on the sphere and notice that you get two different polygons representing the sphere.*

4 All Polygons give connected sum of tori and projective planes

This section is devoted to proving the following theorem.

Theorem 2. *Let P be an identification polygon representing a surface S. Then S is homeomorphic to either a sphere or a finite connected sum of tori and projective planes.*

We prove this theorem by induction on the number of sides $2n$ of the polygon. Note that in Problem 8 we have shown the base case ($n = 1$) of our induction. So assume that every $2n$-gon satisfies the theorem, and let P be a $(2n+2)-gon$. There are a few cases.

Problem 11. *Show that if P has two consecutive sides with the same letter, then you win (regardless of the relative orientations of the two consecutive sides).*

Problem 12. *Assign a word $w(P)$ to the polygon P by reading its sides counterclockwise, and barring the letters where the edge is pointing in the opposite direction. Prove that if in the word the same letter appears both times unbarred (or barred), then you can make a cut and regluing of the polygon that reduces you to the previous exercise.*

Problem 13. *Assume that in the word $w(P)$ all letters appear once barred and once unbarred. Then cut off a strip in such a way that two sides of the strips are sides of the polygon with the same letter. When you identify this side, what does your strip become? Now use the induction hypothesis applied to the remaining part of the polygon (note: there are two cases to consider) and conclude the proof.*

5 Euler Characteristic

Problem 14. *Count the number of vertices (V), edges (E) and faces (F) for:*

- *a tetrahedron;*
- *a cube;*
- *a convex polyhedron of your choice.*

Now compute the following sum:

$$V - E + F$$

What happens?

In topology all convex polyhedra are just **spheres**, and we can think of edges and vertices as a graph on the sphere.

Hope: maybe the above number does not depend on the graph you put on a sphere!

Problem 15. *Show that the above hope is too much to ask for. Find (simple) graphs on the sphere for which the above count is not* 2.

Fix: A **good graph** (on ANY surface, not just a sphere) is a graph such that:

1. there is a vertex any time two edges intersect.
2. there is a vertex at each end of every edge.
3. the complement of the graph is homeomorphic to a disjoint union of discs.

Problem 16. *Put a good graph on a surface. What happens to the count*

$$V - E + F$$

when:

- *you add a vertex in the middle of an edge?*
- *you add an edge connecting two vertices?*
- *you add a vertex in the middle of a face and an edge to connect to an existing vertex?*

Use this to prove that this count is independent of the good graph you put on the surface.

We have therefore just defined a **topological invariant**, called **Euler characteristic**. Now let us use it !

Problem 17. *What is the Euler characteristic of:*

- *the torus?*
- *the projective plane?*

Problem 18. *How does the Euler characteristic behave with respect to connected sums? I.e., if you know the Euler characteristic for S_1 and S_2, what is the Euler characteristic for $S_1 \sharp S_2$?*

Problem 19. *What is the Euler characteristic of:*

- *a connected sum of g tori?*
- *a connected sum of m projective planes?*

Problem 20. *Is the Euler characteristic sufficient to "tell surfaces apart"? In other words, is it posssible for two non-homeomorphic surfaces to have the same Euler characteristic?*

6 Orientability

Orientability is a fascinating topological invariant. It is related to various folklore facts such as that a Mobius strip has only one side, or a Klein bottle has no "inside" or "outside". Unfortunately, it is pretty difficult to define orientability in any sort of generality in an elementary way. For surfaces, we can actually take a shortcut and give an elementary, albeit a bit *ad hoc* definition. Look forward to revisiting this concept in subsequent math classes!

For now, a surface is **orientable** if it contains an open subset homeomorphic to a Mobius strip.

Problem 21. *Prove that orientability is a topological invariant: if $X \cong Y$, then X is orientable if and only if Y is.*

Problem 22. *Show that the projective plane is not orientable.*

Problem 23. *Devise a method to decide whether the surface represented by a polygon is orientable or not by looking at the polygon.*