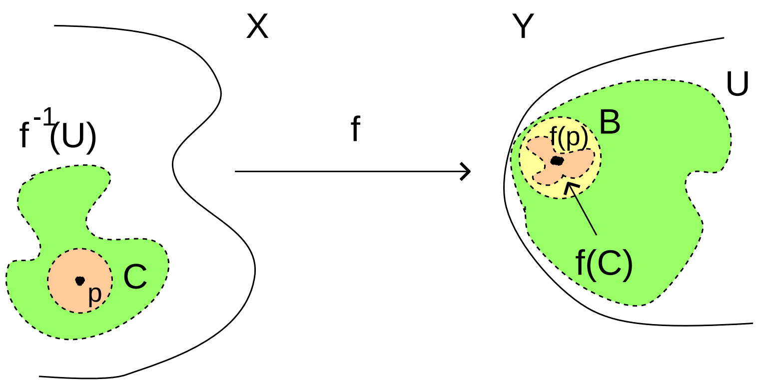
Solutions to homework 1

Question 1: *prove that a function $f : X \longrightarrow Y$ is continuous (calculus style) if and only if the preimage of any open set in Y is open in X.*

PROOF:

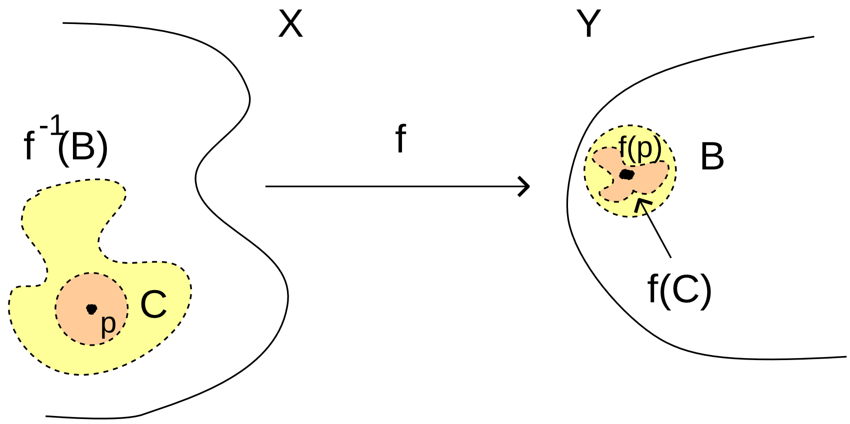


First, assume that f is a continuous function, as in calculus; let U be an open set in Y, we want to prove that $f^{-1}(U)$ is open in X.

If p is a point in $f^{-1}(U)$, we must show there is a little open ball around p that is all contained in $f^{-1}(U)$.

But $f(p) \in U$ which is an open set, so there exists a ball B of radius r centered at $f(p)$ and all contained in U.

Continuity calculus style tells us that provided that we take a small enough radius, there is a ball C around p such that $f(C)$ is contained in B, and hence in U. Which means that C is all contained in $f^{-1}(U)$. So we are done with one side of the proof.



Now assume that for any open set in Y, its preimage via f is open. We want to show that f is a continuous function. Let p be a point in X, $f(p)$ the corresponding image in Y.

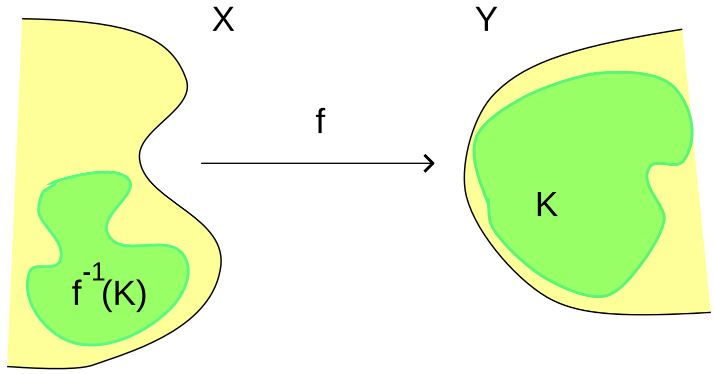
To show that f is continuous at p we must show that, given a ball B of radius ε around $f(p)$, there exists a ball C whose image is entirely contained in B.

But B in particular is an open set. Therefore $f^{-1}(B)$ is open. Therefore p is an interior point for $f^{-1}(B)$: there is a little ball C centered at p contained in $f^{-1}(B)$.

This implies that $f(C)$ is contained in B, which is what we needed to show.

Question 2: *prove that a function $f : X \longrightarrow Y$ is continuous (calculus style) if and only if the preimage of any closed set in Y is closed in X.*

Proof: We want to exploit the previous exercise, and the fact that the complement of an open set is closed.



Assume f is continuous.

Let K be any closed set in Y.

Then $Y \setminus K$ is open.

Then $f^{-1}(Y \setminus K)$ is open by exercise 1.

But $f^{-1}(Y \setminus K) = X \setminus f^{-1}(K)$.

Hence $f^{-1}(K)$ is closed.

Now assume the preimage of any closed set is closed.

Let U be any open set in Y.

$Y \setminus U$ is closed.

Hence $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed.

Which implies that $f^{-1}(U)$ is open. Hence the preimage of any open set is open, and f is continuous by exercise 1.