The Fundamental Group of the Circle

Renzo’s math 472

We sketch a proof of the computation of the fundamental group of the circle. This is on the one hand a very intuitive and easy to believe result. On the other hand it is reasonably sophisticated to prove. Here goes the statement.

Theorem 1.

\[ \Pi_1(S^1) \cong \mathbb{Z} \]

1 The Infinite Rotini

The key player in the proof of this statement is the infinite rotini, i.e. the continuous function:

\[ R : \mathbb{R} \to S^1 \]

declared by

\[ R(y) = e^{2\pi i y} \]

Note: the following are elementary yet crucial properties of \( R \).

1. \( R \) is surjective.

2. For any \( x \in S^1 \), the preimage \( R^{-1}(x) = \{ \ldots , y_n, \ldots \} \) is a countable set, naturally (but not canonically!) in bijection with \( \mathbb{Z} \).

3. For every preimage \( y_n \) there is an open set \( U_{y_n} = (y_n - 1/2, y_n + 1/2) \) such that

\[ R|_{U_n} : U_n \to R(U_n) \]

is a homeomorphism.

Sidenote: \( R \) is an example of a covering map. Every time we have a function \( f : X \to Y \) with properties 1. and 3. we say that \( f \) is a covering map or that \( Y \) is a covering space of \( X \). This is the beginning of a beautiful story... that will be left for another time.
2 Lifting

Given a function \( f : X \to S^1 \) a lifting of \( f \) is a function \( \tilde{f} : X \to \mathbb{R} \) such that:

\[
f = R \circ \tilde{f}.
\]

What is crucial to us is that if \( X \) is either an interval or a square, then:

1. A lifting always exists.
2. Once you specify the image of just one point for \( \tilde{f} \), then the lifting is unique.

The proof of this statement is an excellent exercise for you to think about. You have to exploit property 3 to lift locally (locally \( R \) is a homeo and hence it has an inverse function!). Then you use compactness of the interval and of the square to show that you can lift globally.

3 A silly but useful way to think of \( \mathbb{Z} \)

Denote by \( \Gamma \) the group defined as follows:

**set** homotopy equivalence classes (relative to the endpoints) of paths starting at 0 and ending at some integer.

**operation** composing two paths \( \gamma_1 \) and \( \gamma_2 \) is defined as follows: go along \( \gamma_1 \) twice as fast. Say that the endpoint of \( \gamma_1 \) is \( n \). Then translate \( \gamma_2 \) by \( n \) and go along that path twice as fast. Here is in math notation:

\[
\gamma_1 \star \gamma_2(t) := \begin{cases} 
\gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\
\gamma_2(2t-1) + \gamma_1(1) & \frac{1}{2} \leq t \leq 1
\end{cases}
\]

Since \( \mathbb{R} \) is contractible, it is easy to see that there is only one equivalence class of paths for any given integer endpoint. Upon \( \varepsilon \) more thought it should be reasonably evident that the operation we defined corresponds to additions of integers. Therefore our mystery group \( \Gamma \) is just a funny way of talking about \((\mathbb{Z}, +)\).

4 A natural isomorphism

Now we have natural ways to go from \( \Gamma \) to \( \Pi_1(S^1) \) and back:

**Projection** Given a path \( \gamma \in \Gamma \), \( R \circ \gamma \) is a loop in \( S^1 \).

**Lifting** Given a loop \( \alpha \in \Pi_1(S^1) \), the (unique) lifting \( \tilde{\alpha} \) of \( \alpha \) starting at 0 is a path in \( \Gamma \).
The major issue here is to show that the map \textit{Lifting} is well defined (or if you want, that \textit{Projection} is injective). But this is a consequence of the fact that continuous maps from squares lift as well! Homotopies of paths are in particular maps from a square, and therefore if two loops downstairs are homotopic, so are their lifts, and this in particular implies that their endpoints agree.

The final thing to check is that \textit{Lifting} and \textit{Projection} are group homomorphisms and that they are inverses of each other. And that’s another good exercise for you!

5 A generalization

There is a nice generalization of this theorem that goes as follows.

\textbf{Theorem 2.} Let $X$ be a contractible space and $G$ be a finite group acting on $X$ in such a way that the quotient map

$$X \to X/G$$

is a covering map. Then

$$\Pi_1(X) \cong G$$

Note that in particular this applies to some of our old friends:

1. $\mathbb{P}^2 = S^2/\mathbb{Z}_2$.
2. $T = \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$