The Fundamental Group of the Circle

Renzo's math 472

We sketch a proof of the computation of the fundamental group of the circle. This is on the one hand a very intuitive and easy to believe result. On the other hand it is reasonably sophisticated to prove. Here goes the statement.

Theorem 1.

$$\Pi_1(S^1) \cong \mathbb{Z}$$

1 The Infinite Rotini

The key player in the proof of this statement is the *infinite rotini*, i.e. the continuous function:

$$R : \mathbb{R} \to S^1$$

defined by

$$R(y) = e^{2\pi i y}.$$

Note: the following are elementary yet crucial properties of R.

1. R is surjective.
2. For any $x \in S^1$, the preimage $R^{-1}(x) = \{\ldots, y_n, \ldots\}$ is a countable set, naturally (but not canonically!) in bijection with $\mathbb{Z}$.
3. For every preimage y_n there is an open set $U_{y_n} = (y_n - 1/2, y_n + 1/2)$ such that

   $$R_{|U_n} : U_n \to R(U_n)$$

   is a homeomorphism.

Sidenote: R is an example of a *covering map*. Every time we have a function $f : X \to Y$ with properties 1. and 3. we say that f is a covering map or that Y is a *covering space* of X. This is the beginning of a beautiful story... that will be left for another time.

2 Lifting

Given a function $f : X \to S^1$ a *lifting* of f is a function $\tilde{f} : X \to \mathbb{R}$ such that:

$$f = R \circ \tilde{f}.$$

What is crucial to us is that if X is either an interval or a square, then:

1. A lifting always exists.

2. Once you specify the image of just one point for $\tilde{f}$, then the lifting is unique.

The proof of this statement is an excellent exercise for you to think about. You have to exploit property 3. to lift locally (locally R is a homeo and hence it has an inverse function!). Then you use compactness of the interval and of the square to show that you can lift globally.

3 A silly but useful way to think of $\mathbb{Z}$

Denote by Γ the group defined as follows:

set homotopy equivalence classes (relative to the enpoints) of paths starting at 0 and ending at some integer.

operation composing two paths γ_1 and γ_2 is defined as follows: go along γ_1 twice as fast. Say that the endpoint of γ_1 is n. Then translate γ_2 by n and go along that path twice as fast. Here is in math notation:

$$\gamma_1 \star \gamma_2(t) := \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t-1) + \gamma_1(1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

Since $\mathbb{R}$ is contractible, it is easy to see that there is only one equivalence class of paths for any given integer endpoint. Upon ε more thought it should be reasonably evident that the operation we defined corresponds to additions of integers. Therefore our mystery group Γ is just a funny way of talking about $(\mathbb{Z}, +)$.

4 A natural isomorphism

Now we have natural ways to go from Γ to $\Pi_1(S^1)$ and back:

Projection Given a path $\gamma \in \Gamma$, $R \circ \gamma$ is a loop in S^1.

Lifting Given a loop $\alpha \in \Pi_1(S^1)$, the (unique) lifting $\tilde{(\alpha)}$ of α starting at 0 is a path in Γ.

The major issue here is to show that the map *Lifting* is well defined (or if you want, that *Projection* is injective). But this is a consequence of the fact that continuos maps from squares lift as well! Homotopies of paths are in particular maps from a square, and therefore if two loops downstairs are homotopic, so are their lifts, and this in particular implies that their endpoints agree.

The final thing to check is that *Lifting* and *Projection* are group homomorphisms and that they are inverses of each others. And that's another good exercise for you!

5 A generalization

There is a nice generalization of this theorem that goes as follows.

Theorem 2. *Let X be a contractible space and G be a finite group acting on X in such a way that the quotient map*

$$X \to X/G$$

is a covering map. Then

$$\Pi_1(X) \cong G$$

Note that in particular this applies to some of our old friends:

1. $\mathbb{P}^2 = S^2/\mathbb{Z}_2$.
2. $T = \mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$