Project: Construction of the Fundamental Group

Renzo's math 472

This worksheet is designed to lead you to define and discover our first functor: the fundamental group.

1 Definitions

First of all, let us recall some basic definitions:

1.1 Category theory

A **category** $\mathcal{C}$ is the datum of:

1. a set of objects, usually denoted $Ob(\mathcal{C})$.
2. for each pair of objects $X, Y \in Ob(\mathcal{C})$, a set of morphisms, usually denoted $Hom(X, Y)$.
3. the notion of composition of morphisms, i.e. for every triple of objects X, Y, Z a function $Hom(X, Y) \times Hom(Y, Z) \to Hom(X, Z)$.

Further we require composition of morphisms to be associative and that for each object X we have a distinguished morphism $Id_X \in Hom(X, X)$ which is neutral with respect to composition.

A (covariant) **functor** is a morphism of categories. In other words, given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is the datum of:

1. a function $\mathcal{F} : Ob(\mathcal{C}_1) \to Ob(\mathcal{C}_2)$.
2. for any pair of objects X, Y a function $\mathcal{F} : Hom(X, Y) \to Hom(\mathcal{F}(X), \mathcal{F}(Y))$.

with the natural requirements that identity and compositions are respected, i.e.:

1. $$\mathcal{F}(Id_X) = Id_{\mathcal{F}(X)}$$
2. $$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

1.2 Homotopy theory

homotopy of functions: Let $f, g : X \to Y$ be continuous functions. Then $f \sim g$ (read f is homotopic to g) if there is a continuous function

$$H : X \times I \to Y$$

such that

$$H(x, 0) = f(x), \qquad H(x, 1) = g(x).$$

H is called a homotopy between f and g.

homotopy relative to a subspace: Let $A \subset X$ and $f, g : X \to Y$ be continuous functions such that for any $a \in A, f(a) = g(a)$. Then $f \sim_A g$ (read f is homotopic to g relative to A) if there is a continuous function

$$H : X \times I \to Y$$

such that

$$H(x, 0) = f(x), \qquad H(x, 1) = g(x)$$

and

$$H(a, t) = f(a) = g(a)$$

for all $a \in A, t \in I$.

homotopy equivalence of spaces: Two topological spaces X and Y are homotopy equivalent if there exist functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \sim Id_X$ and $f \circ g \sim Id_Y$.

contractible spaces: Any space which is homotopy equivalent to a point is called **contractible**.

OK, now we are ready to start constructing the fundamental group.

2 The fundamental group

The fundamental group is a functor

$$\Pi_1 : \mathcal{PT} \to \mathcal{G}$$

from the category of pointed topological spaces to the category of groups.

Note that the category of pointed topological spaces is just a slight variant of our familiar category of topological spaces. Objects are pairs (X, x_0), where X is a topological space, and $x_0 \in X$ is a point of X. Morphisms between (X, x_0) and (Y, y_0) are continuous functions $f : X \to Y$ such that $f(x_0) = y_0$.

2.1 The function on objects

Given a pointed topological space (X, x_0), the fundamental group $\Pi_1(X, x_0)$ is constructed as follows.

elements the elements are equivalence classes of loops based at x_0. A loop based at x_0 is a continuous function $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x_0$. Two loops α and β are considered equivalent if they are homotopic relative to their endpoints.

operation the operation of composition is just concatenation of loops. In other words:

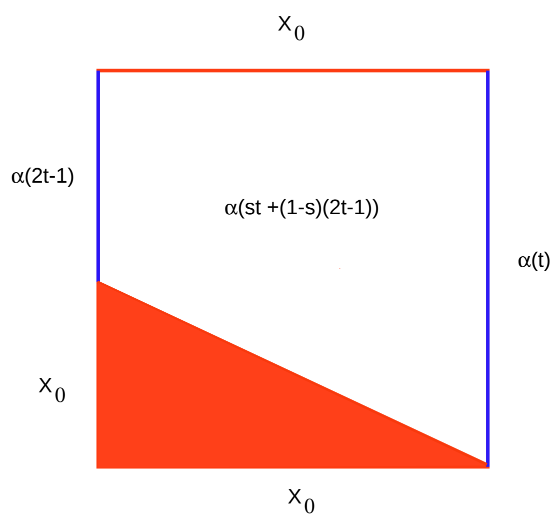
$$\alpha \star \beta(t) := \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Problem 1. *Check that we have indeed defined a group. You must prove the operation is well defined, associative. That there is an identity element, and that each element has an inverse.*

All of these statements can be proven by pictures. Learn to describe a homotopy of loops in terms of a diagram on a square. For example, suppose that we want to prove that the constant loop ε_{x_0} is the identity element in the fundamental group. We then need to show

$$\alpha \sim \varepsilon_{x_0} \star \alpha \sim \alpha \star \varepsilon_{x_0}$$

A homotopy between the two above loops is a function $H : I \times I \to X$, that can be represented by the following diagram:



We read the diagram as follows. The vertical left hand side of the square is the loop $\varepsilon_{x_0} \star \alpha$. The vertical right hand side of the square is α. The square is a homotopy between the two loops. Everywhere you see red is mapped to x_0. The top white trapeze realizes a continuous morphing of "walking around α twice as fast" into "walking around α at regular speed". In this case I have even written down the function in term of s and t. This is a little confusing, since you should think of both t and s as time. The time t is the time along which you walk along the loop. The time s is the time along which you deform your way of walking around the loop.

2.2 The function on morphisms

Now, given a continuous function $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$, we can define a corresponding group homomorphism:

$$\Pi_1(f) : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)$$

$$[\alpha] \mapsto [f \circ \alpha]$$

Problem 2. *Check that indeed $\Pi_1(f)$ is a group homomorphism.*

Problem 3. *Check that Π_1 is indeed a functor between the two above categories.*

3 Some observations about the Fundamental Group

In this section we find some interesting properties of the fundamental group. We already know that the fact that Π_1 is a functor guarantees that it is a topological invariant. In fact, we show that is a homotopy invariant. I.e. in two spaces are homotopy equivalent, them they have isomorphic fundamental group.

Problem 4. *Let $f, g : (X, x_0) \to (Y, y_0)$ be two functions that are homotopic relative to x_0. Then show that*

$$\Pi_1(f) = \Pi_1(g)$$

As a consequence, show that if (X, x_0) and (Y, y_0) are homotopy equivalent relative to the basepoints (figure out what this means!), then

$$\Pi_1(X, x_0) \cong \Pi_1(Y, y_0)$$

What is the fundamental group of a contractible space?

Finally we observe that if a space is path connected, then the choice of the special point is somewhat irrelevant.

Problem 5. *Show that if X is path connected, then for any two points x_0 and $x_1 \in X$ we have*

$$\Pi_1(X, x_0) \cong \Pi_1(X, x_1)$$

This is why one often forgets about the base points and just talks about "the fundamental group of X". However note the subtlety that there is not a unique (or a canonical) isomorphism. You at some point are making a choice. Can you see when?