1 Definitions

First of all, let us recall some basic definitions:

1.1 Category theory

A category $\mathcal{C}$ is the datum of:

1. a set of objects, usually denoted $\text{Ob}(\mathcal{C})$.

2. for each pair of objects $X,Y \in \text{Ob}(\mathcal{C})$, a set of morphisms, usually denoted $\text{Hom}(X,Y)$.

3. the notion of composition of morphisms, i.e. for every triple of objects $X,Y,Z$ a function $\text{Hom}(X,Y) \times \text{Hom}(Y,Z) \to \text{Hom}(X,Z)$.

Further we require composition of morphisms to be associative and that for each object $X$ we have a distinguished morphism $\text{Id}_X \in \text{Hom}(X,X)$ which is neutral with respect to composition.

A (covariant) functor is a morphism of categories. In other words, given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ a functor $\mathcal{F}: \mathcal{C}_1 \to \mathcal{C}_2$ is the datum of:

1. a function $\mathcal{F}: \text{Ob}(\mathcal{C}_1) \to \text{Ob}(\mathcal{C}_2)$.

2. for any pair of objects $X,Y$ a function $\mathcal{F}: \text{Hom}(X,Y) \to \text{Hom}(\mathcal{F}(X),\mathcal{F}(Y))$.

with the natural requirements that identity and compositions are respected, i.e.:

1. $\mathcal{F}(\text{Id}_X) = \text{Id}_{\mathcal{F}(X)}$

2. $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

This worksheet is designed to lead you to define and discover our first functor: the fundamental group.
1.2 Homotopy theory

**Homotopy of functions:** Let \( f, g : X \to Y \) be continuous functions. Then \( f \sim g \) (read \( f \) is homotopic to \( g \)) if there is a continuous function

\[
H : X \times I \to Y
\]

such that

\[
H(x, 0) = f(x), \quad H(x, 1) = g(x).
\]

\( H \) is called a homotopy between \( f \) and \( g \).

**Homotopy relative to a subspace:** Let \( A \subset X \) and \( f, g : X \to Y \) be continuous functions such that for any \( a \in A, f(a) = g(a) \). Then \( f \sim_A g \) (read \( f \) is homotopic to \( g \) relative to \( A \)) if there is a continuous function

\[
H : X \times I \to Y
\]

such that

\[
H(x, 0) = f(x), \quad H(x, 1) = g(x)
\]

and

\[
H(a, t) = f(a) = g(a)
\]

for all \( a \in A, t \in I \).

**Homotopy equivalence of spaces:** Two topological spaces \( X \) and \( Y \) are homotopy equivalent if there exist functions \( f : X \to Y \) and \( g : Y \to X \) such that \( g \circ f \sim Id_X \) and \( f \circ g \sim Id_Y \).

**Contractible spaces:** Any space which is homotopy equivalent to a point is called **contractible**.

OK, now we are ready to start constructing the fundamental group.

2 The fundamental group

The fundamental group is a functor

\[
\Pi_1 : \mathcal{PT} \to \mathcal{G}
\]

from the category of pointed topological spaces to the category of groups.

Note that the category of pointed topological spaces is just a slight variant of our familiar category of topological spaces. Objects are pairs \((X, x_0)\), where \( X \) is a topological space, and \( x_0 \in X \) is a point of \( X \). Morphisms between \((X, x_0)\) and \((Y, y_0)\) are continuous functions \( f : X \to Y \) such that \( f(x_0) = y_0 \).
2.1 The function on objects

Given a pointed topological space \((X, x_0)\), the fundamental group \(\Pi_1(X, x_0)\) is constructed as follows.

**elements** the elements are equivalence classes of loops based at \(x_0\). A loop based at \(x_0\) is a continuous function \(\alpha : I \to X\) such that \(\alpha(0) = \alpha(1) = x_0\). Two loops \(\alpha\) and \(\beta\) are considered equivalent if they are homotopic relative to their endpoints.

**operation** the operation of composition is just concatenation of loops. In other words:

\[
\alpha \ast \beta(t) := \begin{cases} 
\alpha(2t) & 0 \leq t \leq 1/2 \\
\beta(2t - 1) & 1/2 \leq t \leq 1 
\end{cases}
\]

**Problem 1.** Check that we have indeed defined a group. You must prove the operation is well defined, associative. That there is an identity element, and that each element has an inverse.

All of these statements can be proven by pictures. Learn to describe a homotopy of loops in terms of a diagram on a square. For example, suppose that we want to prove that the constant loop \(\varepsilon_{x_0}\) is the identity element in the fundamental group. We then need to show

\[
\alpha \sim \varepsilon_{x_0} \ast \alpha \sim \alpha \ast \varepsilon_{x_0}
\]

A homotopy between the two above loops is a function \(H : I \times I \to X\), that can be represented by the following diagram:

We read the diagram as follows. The vertical left hand side of the square is the loop \(\varepsilon_{x_0} \ast \alpha\). The vertical right hand side of the square is \(\alpha\). The
square is a homotopy between the two loops. Everywhere you see red is mapped to \( x_0 \). The top white trapeze realizes a continuous morphing of “walking around \( \alpha \) twice as fast” into “walking around \( \alpha \) at regular speed”. In this case I have even written down the function in term of \( s \) and \( t \). This is a little confusing, since you should think of both \( t \) and \( s \) as time. The time \( t \) is the time along which you walk along the loop. The time \( s \) is the time along which you deform your way of walking around the loop.

2.2 The function on morphisms

Now, given a continuous function \( f : (X, x_0) \to (Y, y_0) \) such that \( f(x_0) = y_0 \), we can define a corresponding group homomorphism:

\[
\Pi_1(f) : \Pi_1(X, x_0) \to \Pi_1(Y, y_0)
\]

\([\alpha] \mapsto [f \circ \alpha]\)

**Problem 2.** Check that indeed \( \Pi_1(f) \) is a group homomorphism.

**Problem 3.** Check that \( \Pi_1 \) is indeed a functor between the two above categories.

3 Some observations about the Fundamental Group

In this section we find some interesting properties of the fundamental group. We already know that the fact that \( \Pi_1 \) is a functor guarantees that it is a topological invariant. In fact, we show that is a homotopy invariant. I.e. in two spaces are homotopy equivalent, then they have isomorphic fundamental group.

**Problem 4.** Let \( f, g : (X, x_0) \to (Y, y_0) \) be two functions that are homotopic relative to \( x_0 \). Then show that

\[
\Pi_1(f) = \Pi_1(g)
\]

As a consequence, show that if \((X, x_0)\) and \((Y, y_0)\) are homotopy equivalent relative to the basepoints (figure out what this means!), then

\[
\Pi_1(X, x_0) \cong \Pi_1(Y, y_0)
\]

What is the fundamental group of a contractible space?

Finally we observe that if a space is path connected, then the choice of the special point is somewhat irrelevant.
Problem 5. Show that if $X$ is path connected, then for any two points $x_0$ and $x_1 \in X$ we have

$$\pi_1(X, x_0) \cong \pi_1(X, x_1)$$

This is why one often forgets about the base points and just talks about “the fundamental group of $X$”. However note the subtlety that there is not a unique (or a canonical) isomorphism. You at some point are making a choice. Can you see when?