

Project 4: Degree of Maps from a Sphere to Itself

Renzo's math 571

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Throughout this worksheet we study maps for an n -dimensional sphere to itself:

$$f : S^n \rightarrow S^n.$$

The notion of degree we introduce is the right way to formalize the idea of “generic number of preimages” for a map. This takes care of redundancies, and for example, for a map from a circle to a circle, we recover the same notion of degree defined via the fundamental group. But let us go straight to the definition.

Definition 1. *Given a map f as above, then we get the homomorphism:*

$$f_* : H_n(S^n) \rightarrow H_n(S^n).$$

Since we know that the n -th homology group of S^n is just \mathbb{Z} , f_ is just multiplication by an integer d , that we define to be the **degree** of f .*

Now for a bunch of properties that you will have lots of fun proving.

Identity What is the degree of the identity function? Prove it!

Non-surjective functions Prove that if a function is not surjective, then its degree is 0. Show that the converse is not true.

Composition How does the degree behave under composition of functions (I.e. what is the degree of gf in terms of the degrees of f and g)?

Homotopy equivalence If f is a homotopy equivalence, what possible degree can it have?

Antipodal map What is the degree of the antipodal map $Ant(x) = -x$? When can the antipodal map be homotopic to the identity (Note, at this point I am not asking you when it is, only when it has a chance to be!)?

Fixed points Show that if a function has no fixed points, it is homotopic to the antipodal map.

And now for a couple cool theorems...

Theorem 1 (Family of hairy ball theorems). S^n has a continuous never vanishing vector field if and only if n is odd.

In particular, you can't comb a hairy ball! To prove this theorem follow these hints:

1. Note that you can assume that the vectors have all unit length.
2. Think of a vector as pointing you in a particular direction, and to start "walking in that direction".
3. What the previous point is actually asking you to do is to construct a homotopy between the identity function and the antipodal map.
4. What can you conclude about the degree of the two functions?

Theorem 2 (Can't cover much but projective spaces). If n is even, show that the only group that acts freely on S^n is $\mathbb{Z}/2\mathbb{Z}$.

OK, how do we see something like this...

1. Recall that a G action defines a group homomorphism $\varphi : G \rightarrow \text{Homeo}(S^n)$.
2. Degree, on the other hand, gives you a group homomorphism $\text{deg} : \text{Homeo}(S^n) \rightarrow ??$. (This is not a typo, I just would like you to fill in the second group...)
3. Compose the two homomorphisms above to get a homomorphism from G to the final mystery group. Call Φ this composition.
4. Remember that a free action in particular implies that the homeomorphism corresponding to any $g \neq 1$ does not have any fixed points.
5. What does this tell you about the image via Φ of any non-trivial element of g ? What does this tell you about $\text{Ker}(\Phi)$? What can G be?
6. No display a free $\mathbb{Z}/2\mathbb{Z}$ -action on an even sphere.

0.1 Euler Characteristic

If X is a finite dimensional CW complex, we define the notion of Euler characteristic to generalize the notion we had for surfaces.

Definition 2. Let X be a finite dimensional CW complex with c_n n -dimensional cells. The **Euler characteristic** of X is

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n c_n$$

Problem 1. Check that equivalently one can define:

$$\chi(X) = \sum_{n=0}^{\infty} (-1)^n \text{rk}(H_n(X))$$

The proof of this problem follows from linear algebra. Remember that if you have a linear map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, then $n = \dim(\text{Ker}(f)) + \dim(\text{Im}(f))$.

Problem 2. Compute:

- $\chi(S^n)$
- $\chi(\mathbb{R}P^n)$
- $\chi(\mathbb{C}P^n)$

Problem 3. Use the Mayer-Vietoris sequence to inductively compute the homology groups of spheres.