Project 3: The Snake Lemma and the Long Exact Sequence in Homology

Renzo's math 571

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This project develops a fundamental homological algebra tool for us: short exact sequences of complexes give rise to long exact sequences in homology. When this tool is applied in a geometric context, it allows to relate various homology groups.

The main statement we want to prove is the following.

Theorem 1. *Let*

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0 \qquad (1)$$

be a short exact sequence of complexes of abelian groups (or R-modules, if you prefer, with R a commutative group). This induces a long exact sequence in homology:

$$\ldots \to H_k(A_\bullet) \xrightarrow{f_*} H_k(B_\bullet) \xrightarrow{g_*} H_k(C_\bullet) \xrightarrow{\delta} H_{k-1}(A_\bullet) \to \ldots \qquad (2)$$

The key ingredient for the proof is to define the connecting homomorphism δ.

Problem 1 (Snake Lemma)**.** *Given (1), there is a canonical way to define a group homomorphism:*

$$\delta_k : H_k(C_\bullet) \to H_{k-1}(A_\bullet)$$

Given an element $c \in ker\partial \subseteq C_n$, obtain in a natural way an element $a \in A_{n-1}$. This is not a function, but rather a correspondence, since you can obtain different a's starting from the same c. However show that this correspondence induces a well defined function at the level of homology. There are a few things to show here:

1. $a \in ker\partial$
2. any two a's that you may associate to the same c differ by a boundary, and therefore represent the same homology class.

3. if c is a boundary, then the associated homology class a is the zero class.

Problem 2. *Having defined the connecting homomorphism, now you need to show that the sequence (2) is exact.*

There are six things to check:

1. $Im(f_*) \subseteq Ker(g_*)$(aka $g_* f_* = 0$).
2. $Im(f_*) \supseteq Ker(g_*)$.
3. $Im(g_*) \subseteq Ker(\delta_*)$(aka $\delta g_* = 0$).
4. $Im(g_*) \supseteq Ker(\delta_*)$.
5. $Im(\delta) \subseteq Ker(f_*)$(aka $f_* \delta = 0$).
6. $Im(\delta) \supseteq Ker(f_*)$.