Projects

Renzo's math 570

1 Compactifications

Compact spaces are very nice for many different reasons, most of which will only become evident much further along in your mathematical journey. Given a non compact space X, one would sometimes like to construct a homeomorphism from X to an open dense set of a compact space $\tilde{X}$. (Informally, this means, adding some points so that X "becomes compact".) $\tilde{X}$ is called a **compactification** of X.

Assume throughout that X is:

- Hausdorff.
- **Locally compact:** this simply means that every point of x is contained in a compact neighborhood.

Define the **one-point compactification** $\tilde{X}$ of X:

as a set: $\tilde{X}$ has precisely one more point that the points of X. We suggestively call this point **infinity**.

$$\tilde{X} = X \cup \{\infty\}$$

topology: we discuss open sets according to whether they contain ∞ or not:

- if $\infty \notin U$, then U is open in $\tilde{X}$ if and only if it is open in X.
- if $\infty \in U$, then U is open in $\tilde{X}$ if and only if its complement is compact.

Problem 1. *Show that for any open set U in $\tilde{X}$, $U \cap X$ is open in X.*

Problem 2. *If X is not compact, then prove that:*

- *X is homeomorphic to an open dense set in $\tilde{X}$.*
- *$\tilde{X}$ is compact.*
- *$\tilde{X}$ is connected.*

- $\tilde{X}$ *is Hausdorff.*

What happens if you start with an X *that is already compact?*

Problem 3. *Show that if* X *and* Y *are homeomorphic, so are their one point compactifications.*

Problem 4. *Show that the one point compactification of the plane (with euclidean topology) is (homeomorphic to) the sphere.*

Problem 5. *What is the one point compactification of:*

- *an open disc in the plane.*
- *a closed disc in the plane minus a point.*
- *an open interval of the real line.*
- *a bunch of open intervals of the real line.*

Problem 6. *Find an (or even better, some) example(s) of two spaces* X *and* Y *that are NOT homeomorphic, but such that their one point compactifications are homeomorphic.*

Problem 7. *Prove that* $\mathbb{P}^2$ *is compact, and that* $\mathbb{P}^2$ *is a compactification of the plane. At this point can you decide whether it is a one-point compactification or not?*