Making New Spaces from Old Ones

Renzo's math 570

In this set of problems we want to illustrate, with some hopefully interesting and useful examples, the strategy discussed in class about how to induce a topology on a set X, given the datum of a topological space (Y, τ_Y) and a set function either from Y to X or from X to Y. The resulting topology on X is called the **induced topology** (induced by the pair (Y, τ_Y)).

1 The philosophy

1.1 $f : X \to (Y, \tau_Y)$

In this case we define the induced topology to be the coarsest topology on X that makes f continuous. Equivalently, the induced topology is composed of all preimages of open sets of Y.

1.2 $X \leftarrow (Y, \tau_Y) : f$

In this case the induced topology is the finest topology on X that makes f continuous. Or the induced topology is generated by all subsets of Y whose preimage is open in X.

2 The subspace topology

If X is a subset of (Y, τ_Y), we can use the inclusion function $i : X \to Y$ to induce a topology on X. In this case open sets of Y are obtained by intersecting open sets in X with Y.

Problem 1. *Consider the closed interval* $[0, 1] \subset (\mathbb{R}, \tau_{eucl})$*. Describe the open sets in the subspace topology.*

3 The product topology

Let (X, τ) and (Y, σ) be two topological spaces. We define a new topological space $X \times Y$, called the **product (space) of X and Y**. We define a natural topology on $X \times Y$, starting from τ and σ, called the **product topology**.

$X \times Y$ **as a set:** the points of $X \times Y$ correspond to ordered pairs (x, y), where $x \in X$ and $y \in Y$,

Recall that you have two natural maps from a product space called the **projections**:

$$\begin{array}{rccc} \pi_X : & X \times Y & \longrightarrow & X \\ & (x,y) & \mapsto & x \end{array}$$

$$\begin{array}{rccc} \pi_Y : & X \times Y & \longrightarrow & Y \\ & (x,y) & \mapsto & y \end{array}$$

According to our philosophy, the **product topology** is the coarsest topology that makes both projection maps continuous.

Problem 2. *Show that the following definition of the product topology is equivalent.*

A basis for the product topology: *a* ***basic open set*** $B \in \beta$ *is a set of the form* $U \times V$*, with* $U \in \tau$ *and* $V \in \sigma$*).* β *is a base for the* ***product topology****.*

Problem 3. *Familiarize yourself with the product topology. Draw some pictures, make some examples, understand what the basic open sets look like, and find some examples of open sets in the product topology that are not basic opens.*

Problem 4. *Show that the cylinder is a product space. Show that the product topology coincides with the induced topology from the euclidean topology in* $\mathbb{R}^3$*.*

4 The quotient topology

If you can define an equivalence relation $\sim$ on the points of a topological space Y, then you get a surjective map to the quotient set:

$$\pi : Y \to Y/\sim$$

The **quotient topology** is the finest topology on the quotient set that makes π continuous. We now discuss a few typical ways to encounter quotient spaces.

4.1 Identification spaces

We can define an equivalence relation on some points of Y just by "declaring" some subsets of points to be equivalent. This has the effect of identifying such points.

Problem 5. *Carefully show that the identification space* $[0,1]/\{0 \sim 1\}$ *is homeomorphic to the circle with euclidean topology.* ***Note:*** *the notation above means the following: on the closed segment* $[0,1]$ *define an equivalence relation where reflexivity and symmetry are implicitely assumed, where the point* 0 *is equivalent to* 1 *and no other relation is imposed.*

4.2 Attaching spaces along a map

Let Y and Z be topological spaces, $W \subset Y$ a subset of Y and suppose you have a continuous function $f : W \to Z$. Then you can generate an equivalence relation on the disjoint union $Y \coprod Z$ by declaring equivalent all points of the form $(w, f(w))$. This has the effect of gluing together points of Y with their corresponding images in Z.

Problem 6. *Let* Y *and* Z *be two copies of the unit closed interval,* $W = \{0,1\}$ *and* $f : \{0,1\} \to Z$ *be the natural inclusion of the two points into the unit interval (*$f(0) = 0$ *and* $f(1) = 1$*). Show that again the space obtained by attaching the two intervals along* f *is homeomorphic to a circle.*

4.3 Orbit spaces

We start here with a simple scenario. Let G be a finite or a countable group (with multiplication read from right to left), and consider it a topological space by giving it the discrete topology (we will call such a creature a **discrete group**). Y is a topological space. A **group action** is a continuous function:

$$\varphi : G \times Y \to Y$$

such that:

1. for any point $y \in Y$, the identity element acts trivially on y: $\varphi(1, y) := 1 \cdot y = y$.
2. for g_1, g_2 any pair of elements of G:

$$g_2 \cdot (g_1 \cdot y) = (g_2 g_1) \cdot y$$

Then you can declare an equivalence relation on Y by declaring $y_1 \sim y_2$ whenever there is a group element that takes one into the other (i.e. there exists $g \in G$ such that $g \cdot y_1 = y_2$. Equivalence classes are called **orbits**, and the topological space obtained by inducing a topology on the quotient set is called the **orbit space**.

Problem 7. *Let* $G = \mathbb{Z}$ *act on the real line by the action*

$$n \cdot x = x + n$$

Check carefully that this is a kosher group action, and verify that the orbit space is (yet again) homeomorphic to the circle.

5 Problems for everyone

5.1 The torus

A **torus** is (informally) the topological space corresponding to the surface of a bagel, with topology induced by the euclidean topology.

Problem 8. *Realize the torus as a product space.*

Problem 9. *Realize the torus as an identification space starting from a square.*

Problem 10. *Realize the torus as a quotient space of the euclidean plane by an appropriate action of the group* $\mathbb{Z} \oplus \mathbb{Z}$

5.2 Cones

For any topological space X we define a new topological space CX called the **cone over** X. CX is defined as an identification space of the product $X \times [0, 1]$, where you identify all points of the form $(x, 1)$ together.

Problem 11. *Draw some pictures and familiarize yourselves with this construction. Why is it called "cone"?*

Problem 12. *Show that the cone over a closed disc is homeomorphic to a closed three dimensional ball, but the cone over an open disc is not homeomorphic to an open ball.*

5.3 Suspensions

The **suspension** SX of X is the identification space obtained from $X \times [-1, 1]$ by the rule:

- all points of the form $(x, -1)$ are identified;
- all points of the form $(x, +1)$ are identified;
- all other points are left alone.

Problem 13. *Show that the suspension* SX *can be obtained by gluing two cones over* X *along a function.*

Problem 14. *What is the suspension of a circle homeomorphic to?*

5.4 Balls and Spheres

Recall the definition of balls and spheres in euclidean space:

- the n-dimensional (closed) ball $\overline{B}^n \subset \mathbb{R}^n$ is the set of $x \in \mathbb{R}^n$ with $||x|| \leq 1$.
- the $(n-1)$-dimensional sphere $S^{n-1} \subset \mathbb{R}^n$ is the set of $x \in \mathbb{R}^n$ with $||x|| = 1$.

Problem 15. *Show that the sphere S^n can be obtained as (i.e. is homeomorphic to) an identification space from the ball $\overline{B}^n$, provided $n > 0$.*

Problem 16. *Show that the ball $\overline{B}^n$ can be obtained as (i.e. is homeomorphic to) the cone CS^{n-1}. Look at project 1 for the definition of a cone.*

5.5 The Projective Line

The projective line is a space parameterizing all lines through the origin in $\mathbb{R}^2$. This means that there is a natural bijection between the set of points of the projective plane and the set of lines through the origin in three dimensional euclidean space.

Problem 17. *Define a natural bijection between the projective line and the quotient space of a circle by the action of a finite group. Show that this gives a homeomorphism between the projective line and a circle.*

5.6 The Projective Plane

The projective plane is a space parameterizing all lines through the origin in $\mathbb{R}^3$. This means that there is a natural bijection between the set of points of the projective plane and the set of lines through the origin in three dimensional euclidean space. The following are some useful ways to visualize the set of points in this space.

$$\mathbb{P}^2 = \frac{\mathbb{R}^3 \setminus \{0\}}{\{(X,Y,Z) = (\lambda X, \lambda Y, \lambda Z)\}} = \frac{\mathit{Sphere}}{\{P = -P\}}$$

There are three natural functions from the plane to $\mathbb{P}^2$:

$$\begin{array}{rccc} \varphi_z : & \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (x,y) & \mapsto & (x:y:1) \end{array}$$

$$\begin{array}{rccc} \varphi_y : & \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (x,z) & \mapsto & (x:1:z) \end{array}$$

$$\begin{array}{rccc} \varphi_x : & \mathbb{R}^2 & \longrightarrow & \mathbb{P}^2 \\ & (y,z) & \mapsto & (1:y:z) \end{array}$$

Problem 18. *Induce a topology on $\mathbb{P}^2$ using our philosophy with respect to the three natural inclusion functions: the finest topology that makes all three inclusion functions continuous.*

Describe this topology, and show that the images $\varphi_x(\mathbb{R}^2)$, $\varphi_y(\mathbb{R}^2)$, $\varphi_z(\mathbb{R}^2)$ become open dense sets of $\mathbb{P}^2$

Problem 19. *Define a natural map from the sphere to $\mathbb{P}^2$. Define a topology on $\mathbb{P}^2$ using our philosophy with respect to this map: the finest topology that makes this map continuous. Show that this topology is the same as the topology defined in the previous problem.*

Problem 20. *Realize the projective plane as a quotient of some space via the action of the cyclic group $\mathbb{Z}/2\mathbb{Z}$.*

Problem 21. *Realize the projective plane as an identification space from a disc.*