Project 2

Renzo's math 571

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This worksheet is supposed to guide you through formally proving that two homotopic functions between topological spaces induce two homotopic morphisms of singular chain complexes. This in turn guarantees that the induced morphisms in homology are the same, and that homology is a homotopy invariant.

Step 1: the Prism

The first step is to identify the product $\Delta_n \times I$ as a formal sum of $(n+1)$ $(n+1)$-simplices (yes, that is not a typo, the first is the number of simplices, the second their dimension).

In $\mathbb{R}^{n+1}$ denote the origin by v_0, the point $(0, \ldots, 0, 1)$ by w_0; v_i denotes the point that has all coordinates 0 except the i-th which is 1, and w_i the point that has 1 only in the i-th and $(n+1)$-th coordinate. Note that the v's naturally identify the set $\Delta_n \times \{0\}$ and the w's the set $\Delta_n \times \{1\}$.

Define an ordering on these points by setting:

$$v_0 < v_1 < \ldots < v_n < w_0 < w_1 < \ldots < w_n.$$

Let $P \in C_{n+1}(\mathbb{R}^{n+1})$ be defined as follows:

$$P := \sum_{i=0}^{n} (-1)^i \langle v_0 \ldots v_i w_i \ldots w_n \rangle.$$

The notation $\langle v_0 \ldots v_i w_i \ldots w_n \rangle$ means the affine map from the standard $(n+1)$ simplex to $\mathbb{R}^{n+1}$ sending the origin to v_0, the first vertex to v_1, ..., the i-th vertex to v_i, the $(i+1)$-th vertex to w_i etc.

Check that the support of P is precisely $\Delta_n \times I$, and that the internal faces appear twice in the formal sum and cancel. Start by working out some low dimensional examples very concretely, and see if you can work up to a general proof. You can look at Hatcher, page 112, for some help.

Step 2: definition of the homotopy

Now assume $f, g : X \to Y$ are two homotopic functions. Recall this means there is a continuous function

$$H : X \times I \to Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

For any n, define

$$H^{sing} : C_n(X) \to C_{n+1}(Y)$$

by setting

$$H^{sing}(\sigma) = H \circ (\sigma \times Id) \circ P$$

Check that H^{sing} is a homotopy between the induced morphism on singular chain complexes. I.e. that for any n,

$$g_n - f_n = H^{sing} \circ \partial_n + \partial_{n+1} \circ H^{sing}.$$